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Towards a Covariant Classification of Nilpotent Four-Qubit States

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Abstract

Quantum Entanglement lies at the heart of some of the strangest phenomena physics has to offer and finds possible applications for example in Quantum Communication and Computation. Yet, a rigorous mathematical formalism that can robustly *quantify* entanglement beyond the two and three-qubit cases is elusive. In the present text we review a rigorous classification of three-qubit entanglement using the formalism of covariant and invariant quantities under the SLOCC equivalence group. This classification is then attempted to be generalised to the case of four-qubit states of nilpotent SLOCC orbits, building on previous work on a subset of these states, and partial success of this ongoing work is presented.

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Rash, thanks for putting up with me.

Für meine Familie.

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1 Introduction

Quantum Entanglement enables us to observe some of the most fascinating, and counter-intuitive (to the classically-minded observer) phenomena physics has to offer. We can use Quantum Entanglement as a resource that allows us to perform classically impossible tasks, such as *Quantum Teleportation*, *Quantum Key Distribution* and *Quantum Computation*.

A lot of progress has been made recently in formalising the mathematical foundations of Quantum Entanglement, enabling more precise definitions of the notions of the *quantity* and the *quality* of entanglement. The deeper questions, rooted in the philosophy of science, of *how* Nature implements entanglement still lack any answer at all.

In this report the rigorous *classification* of entangled states of four-qubit systems is the main focus.

After introducing the necessary preliminaries, we will look at one of the examples of the kind of strange behaviour Quantum Entanglement enables, in a discussion of *Non-Local Games* in section 2.

These are games in which the players can make use of entangled states to improve their chances of winning beyond what is classically possible. Crucially for our discussion, we will also see that there are different ways in which states can be entangled. Not just that qubits in a subset of a given state can be entangled among themselves while leaving another subset separable, even completely non-separable states can be entangled in different ways.

This suggests the need for a *classification* quantifying the *amount* and the *kind* of entanglement contained in a state.

In order to give such a classification, we examine what it means for states to share equivalent entanglement properties or an equivalent amount of entanglement, introducing the notions of *LOCC* and *SLOCC equivalence* in section 3.

For two and three-qubit states an entanglement classification exists, and we will review it in section 4. However, for larger systems it is still lacking. Attempting to systematically quantify the entanglement of nilpotent four-qubit systems under the paradigm of *SLOCC equivalence* using invariant and covariant quantities is the main subject of the present text and is done in section 5. In order to do so, we review the three-qubit classification using *SLOCC covariant* and *invariant* quantities, before turning our attention to the four-qubit case. Our focus will lie on four-qubit states living on *nilpotent orbits* of the *SLOCC equivalence* group, a subset of which has been previously classified in the literature [1]. Generalising the classification of this subset to the complete set of nilpotent four-qubit states is the main purpose of the present text. For separable, tri-separable and bi-separable states this can be done successfully, while for completely non-separable states this work in ongoing.

In the following we will briefly introduce some of the necessary concepts and notations, as well as give some mathematical background information.

1.1 Quantum Bits

In this report we are interested in the study of Quantum Bits, Qubits. They are the fundamental unit when considering Quantum Information, Computation, and Communication, just like the Classical Bit is in the Classical counterparts of these fields. As the name suggests, these are two-level systems which can thus model spin-$\frac{1}{2}$ particles. It is important to note, however, that the physical interpretation of a qubit is not a prerequisite to studying the above topics. The mathematical formalism can stand on its own. The analogy to a physical spin-$\frac{1}{2}$ system facilitates the introduction of the necessary concepts.
Such a system can adopt two possible values, a spin of $\pm\frac{1}{2}$. These can be denoted in various ways, where a spin-up state and a spin-down state as can be written as

$$|1\rangle = |\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |0\rangle = |\downarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.1.1}$$

respectively. The generalisation of the concept of a Qubit to a system with more than two levels is often referred to as a *Qudit*, where the system has d levels. For the case of $d = 3$ one speaks of a *Qutrit*.
The discussion that follows will largely be limited to Qubits.

1.2 The Quantum State

1.2.1 The Single Qubit State

The general unnormalised state of a system consisting of a single qubit can be written

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \ , \tag{1.2.1}$$

where α, β are complex numbers. When describing *physical* states, it is important that the sum of all possible outcomes of measurements of these states will be equal to one. This constrains the individual coefficients, such that

$$|\alpha|^2 + |\beta|^2 = 1. \tag{1.2.2}$$

Hence, the normalised state in equation 1.2.1 can be written as

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\,|1\rangle\right) \ , \tag{1.2.3}$$

where θ, φ are real numbers [2].
Note, moreover, that multiplication by an overall complex phase makes no observable, physical difference, and can be disregarded. Consequently, in (1.2.3) one can ignore the overall complex phase $e^{i\gamma}$. This allows for an interpretation of θ, φ as angles in polar coordinates. The state can thus be plotted on a sphere,

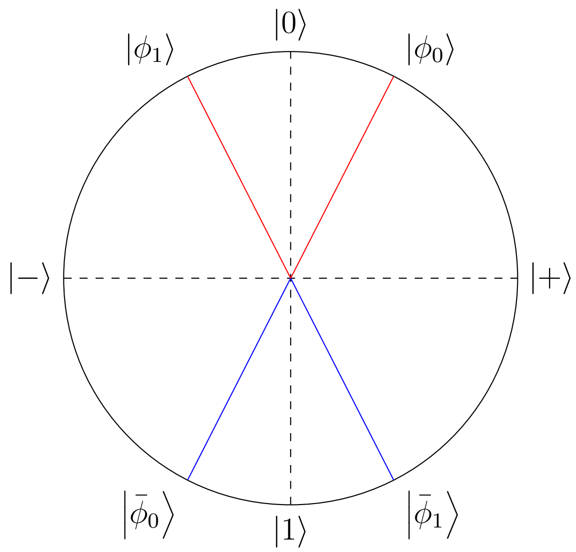

Figure 1: 2d representation of the Bloch Sphere

the so-called Bloch Sphere. A two-dimensional representation of a Bloch Sphere can be seen in figure 1, where the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. The third, omitted direction on the Bloch Sphere contains the states $|i\rangle_{\pm} = \frac{1}{\sqrt{2}}(|0\rangle \pm i\,|1\rangle)$.
Notice that a general state in the two-dimensional representation of the Bloch Sphere can be written [3]

$$\begin{aligned} |\phi_0\rangle &= \cos\left(\frac{\alpha}{2}\right)|0\rangle + \sin\left(\frac{\alpha}{2}\right)|1\rangle \qquad & \left|\bar{\phi}_0\right\rangle &= \sin\left(\frac{\alpha}{2}\right)|0\rangle - \cos\left(\frac{\alpha}{2}\right)|1\rangle\,, \\ |\phi_1\rangle &= \cos\left(\frac{\alpha}{2}\right)|0\rangle - \sin\left(\frac{\alpha}{2}\right)|1\rangle \qquad & \left|\bar{\phi}_1\right\rangle &= \sin\left(\frac{\alpha}{2}\right)|0\rangle + \cos\left(\frac{\alpha}{2}\right)|1\rangle\ . \end{aligned} \tag{1.2.4}$$

This will become important in due course, when we discuss the two-player CHSH game in section 2.1.3.
The Bloch Sphere can be a very helpful representation when considering Quantum Games and Quantum Communication protocols. Using the Bloch Sphere, it can be very easy to identify the overlap of given states and to choose states with a certain overlap to other states, which is advantageous in certain applications where Quantum Entanglement is used as a resource, such as Quantum Teleportation. Examples of these applications are discussed in the following section on Quantum Entanglement.
To summarise, the *unnormalised single qubit* state introduced in equation (1.2.1) describes systems with two complex degrees of freedom, and as such lives in a complex vector space, a Hilbert Space $\mathscr{H} = \mathbb{C}^2$.
The set of *physical single qubit* states is the set of undirected (complex) lines

through the origin, or rays in $\mathscr{H} = \mathbb{C}^2$, since we normalise and disregard overall phases. More precisely, we introduce the equivalence relation $(z_1, z_2) \sim (\tau z_1, \tau z_2)$ where $\tau \in \mathbb{C} \setminus \{0\}$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$, which defines the complex projective space $\mathbb{CP}^1$. Note, $\mathbb{CP}^1 \simeq S^2$ and hence it can be represented conveniently on the Bloch Sphere.

1.2.2 Multi-Qubit States

The discussion in the previous subsection focused on states of a single qubit. In order to consider larger systems, featuring multiple qubits, the *Tensor Product* formalism is introduced here. As an example, combining two qubits using the tensor product proceeds as

$$
\begin{aligned}
|00\rangle &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 1 \otimes \begin{pmatrix}1\\0\end{pmatrix} \\ 0 \otimes \begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \\
|01\rangle &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 1 \otimes \begin{pmatrix}0\\1\end{pmatrix} \\ 0 \otimes \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \\
|10\rangle &= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} 0 \otimes \begin{pmatrix}1\\0\end{pmatrix} \\ 1 \otimes \begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \\
|11\rangle &= \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} 1 \otimes \begin{pmatrix}0\\1\end{pmatrix} \\ 1 \otimes \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.
\end{aligned}
\tag{1.2.5}
$$

More generally, the state of an n-qubit system can be written

$$
|\psi\rangle = \bigotimes_{x=0}^{n-1} (\alpha_x |0\rangle + \beta_x |1\rangle) = \sum_{\omega=0}^{n^2-1} \gamma_\omega \left| \underset{2}{\omega} \right\rangle , \tag{1.2.6}
$$

where $\underset{2}{\omega}$ denotes the decimal value of ω written in binary, and we have introduced the shorthand notation

$$
\bigotimes_{i=0}^{n-1} |A_i\rangle = \underbrace{|A_0\rangle \otimes |A_1\rangle \otimes \cdots \otimes |A_{n-1}\rangle}_{n \text{ times}} = \underbrace{|A_o\rangle |A_1\rangle \ldots |A_{n-1}\rangle}_{n \text{ times}} = \Big| \underbrace{A_0 A_1 \ldots A_{n-1}}_{n \text{ times}} \Big\rangle , \tag{1.2.7}
$$

where $A_i \in \{0, 1\}$.
For example,

$$|0\rangle \otimes |1\rangle \otimes |0\rangle = |0\rangle |1\rangle |0\rangle = |010\rangle \ . \tag{1.2.8}$$

The normalisation condition on (1.2.6) then becomes $\sum_{\omega=0}^{n^2-1} |\gamma_\omega|^2 = 1$.
Extending the discussion from the previous subsection, an n-qubit state lives in the Hilbert Space

$$\mathscr{H} = \mathbb{C}^{2^{\otimes n}} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n-times} . \tag{1.2.9}$$

Physical states, that is rays in $\mathscr{H}$, belong to the complex projective space $\mathbb{P}(\mathbb{C}^{2^{\otimes n}}) \simeq \mathbb{CP}^{2^n-1}$.

1.2.3 Mixed Quantum States

In general, Quantum States cannot always be written in terms of a state vector $|\psi\rangle$. This is the case, for example, when a system is prepared with some uncertainty, where with some probability it was prepared in one state, and with some probability in another. We refer to such a system as being in a *mixed* Quantum State, as opposed to a *pure* one.
The formalism used to express these more general Quantum States is that of the *Density Matrix* or *Density Operator*, and it becomes especially important when talking about these non-pure states.
Suppose a Quantum System is prepared in a certain state probabilistically, for whatever reason, where it is prepared in the state $|\psi_i\rangle$ with probability p_i, and $\{\psi_i, p_i\}$ is an ensemble of pure states.
Then its state is described by a density operator of the form [4]

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle\psi_i| \ . \tag{1.2.10}$$

All other notions that apply to state vectors, such as measurement, unitary evolution, expected values and the post-measurement state also generalise to the density operator. More details are provided in the section on Unitary Evolution and Measurement 1.3.
In fact, the density matrix formalism is the more general formalism, and does not rely on the notion of a state vector for interpretation.
The density operator is a positive operator, for which $\operatorname{Tr} \rho = 1$.
The density matrix of a pure state is simply given by equation (1.2.10) where only one i is non-zero, and $p_i = 1$, in which case $\rho^2 = \rho$, otherwise the state is mixed.
The formalism of density operators also allows us to consider states that feature *classical correlations*. A state is classically correlated if the density matrix can be written as

$$\rho = \sum_{ij} p_{ij} |i\rangle \langle i| \otimes |j\rangle \langle j| \ , \tag{1.2.11}$$

where p_{ij} is a joint probability distribution for i, j. [5]

1.3 Unitary Evolution and Quantum Measurement

The systems described by the above states (1.2.1), (1.2.6) contain their individual qubits in a superposition of the states $|0\rangle$ and $|1\rangle$. Which of these outcomes is obtained when these systems are measured is of course entirely probabilistic. The respective probabilities are dependent on the coefficients α_x and β_x due to the probabilistic collapse of the wavefunction upon measurement. This uncertainty, being a defining property of Quantum Mechanics, lies at the very heart of Quantum Information, Computation, and Communication. It is intimately related to Quantum Entanglement, which will be discussed in due course.

The tools we have at our disposal to manipulate these states, apart from performing measurements, are unitary transformations. These are invertible operations acting on the qubits which preserve probabilities, represented by unitary matrices. Some very commonly used unitary transformations acting on a single qubit are often referred to as *Quantum Gates* in the context of Quantum Computing. There are four unitary and hermitian matrices which play a very important role in performing fundamental operations on single qubits, the three Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ and the Hadamard matrix H. They are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{1.3.1}$$

These matrices represent basic operations on single qubits, making them a representation of single qubit *Quantum Gates*.

In order to analyse the action of the unitaries given in (1.3.1), first notice that each of the matrices has eigenvalues $+1, -1$. The corresponding eigenvectors of σ_z are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{+1}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{-1}$, which correspond to the states $|0\rangle$ and $|1\rangle$. We therefore say that a state written in a basis of $|0\rangle$ and $|1\rangle$ is in the Pauli Z basis, simply Z basis, or computational basis. We see immediately that a Pauli Z acting on a state maps $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto -|1\rangle$, implying that it leaves a state in its eigenbasis invariant (up to a global phase).

Given that the eigenvectors of σ_x are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_{+1}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}_{-1}$, a state written in terms of $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is said to be in the X, Pauli X, or Hadamard basis, and is left invariant by the Pauli X operation (up to a global phase).

Similarly, one notes the eigenvectors of σ_y to be $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}_{+1}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}_{-1}$. States of the form $|i\rangle_\pm = \frac{1}{\sqrt{2}}\left(|0\rangle \pm i\,|1\rangle\right)$ are then in the Y or Pauli Y basis, and left invariant under the Pauli Y operation (up to a global phase).
In a two dimensional plot of the Bloch Sphere, these states are represented as being orthogonal, as can be seen from figure 1.
Generally, the Pauli matrices act as

$$\begin{aligned}
\sigma_x &: \Big\{\, |0\rangle \mapsto |1\rangle\,, \quad |1\rangle \mapsto |0\rangle\,, \quad |i\rangle_+ \mapsto -\,|i\rangle_-\,, \quad |i\rangle_- \mapsto -\,|i\rangle_+ \Big\}\\
\sigma_y &: \Big\{\, |0\rangle \mapsto i\,|1\rangle\,, \quad |1\rangle \mapsto -i\,|0\rangle\,, \quad |+\rangle \mapsto -i\,|-\rangle\,, \quad |-\rangle \mapsto i\,|+\rangle \Big\}\\
\sigma_z &: \Big\{\, |+\rangle \mapsto |-\rangle\,, \quad |-\rangle \mapsto |+\rangle\,, \quad |i\rangle_+ \mapsto |i\rangle_-\,, \quad |i\rangle_- \mapsto |i\rangle_+ \Big\}\,,
\end{aligned} \tag{1.3.2}$$

and the Hadamard is responsible for $H\,|0\rangle = |+\rangle\,, H\,|1\rangle = |-\rangle$ and vice versa, where the inverse operation follows immediately from the unitarity condition $HH^\dagger = \mathbb{1}$.
At this point it should be noted that the Pauli Matrices are also the generators of the Lie Algebra $\mathfrak{su}(2)$, such that we see from the actions of the Pauli matrices in (1.3.2) that $SU(2)$ acts transitively on $\mathbb{CP}^1$. This follows from the interpretation of $SU(2)$ as the double cover of $SO(3)$, the group manifold of which is S^3, such that we have $\mathbb{CP}^1 \simeq \frac{S^3}{U(1)} \simeq \frac{SU(2)}{U(1)}$.
In terms of the density matrix formalism introduced above, unitary evolution acts on the density operator straight-forwardly as $\rho \mapsto U\rho U^\dagger$, where U is a unitary matrix.
In *Quantum Computing* contexts one routinely also encounters unitary operations that act on more than one qubit, such as operations that act on one qubit controlled on the value of another. However, for our purposes single qubit operations are sufficient to have been introduced here.

POVMs

As mentioned at the beginning of this subsection, the other important concept in dealing with Quantum States is Measurement. The most general way of representing Quantum Measurements is by means of the concept of *Positive Operator Valued Measures* (POVMs). A POVM is defined as a set of operators K_μ, such that $\sum_{\mu=1}^m K_\mu^\dagger K_\mu = \mathbb{1}$ for some m, with the probability of a specific measurement outcome being $p_\mu = \mathrm{Tr}\left(K_\mu \rho K_\mu^\dagger\right)$, and upon observing outcome μ the initial state becomes $\rho \to \frac{K_\mu \rho K_\mu^\dagger}{p_\mu}$. [6]

Considering only the case where K_μ is a 1-dimensional projector over a vector $|\mu\rangle$, and the set $\{|\mu\rangle\,, \mu = 1, \ldots, d\}$ form an orthonormal basis, one recovers the formalism of projective ("von Neumann") measurements, for which $p_\mu = \mathrm{Tr}\left(|\mu\rangle\langle\mu|\,\rho\,|\mu\rangle\langle\mu|\right) = \langle\mu|\,\rho\,|\mu\rangle$, and post-measurement $\rho \mapsto \frac{|\mu\rangle\langle\mu|\rho|\mu\rangle\langle\mu|}{p_\mu} = |\mu\rangle\langle\mu|$. [6]

1.4 Spin-Flip Operation

In this section a concept necessary for some of the following, the Spin-Flip Operation, is introduced, closely following the treatment of Sakurai in [7].
In order to define the spin-flip of a state, we first define a general time reversal operator by its property of reversing the direction of angular momentum $\Theta J \Theta^{-1} = -J$.
In order to look at the spin-flip, we are interested in applying this operator to a spin-$\frac{1}{2}$ system. Applying the time-reversal operator to the state vector of a spin-up state with spin in the $\hat{n}$ direction (the direction of $\hat{n}$ being given by the azimuthal angle α and polar angle β), we obtain

$$\Theta\,|\hat{n},+\rangle = e^{-\frac{i}{\hbar}S_z\alpha}e^{-\frac{i}{\hbar}S_y\beta}\Theta\,|+\rangle = \eta\,|\hat{n},-\rangle \ , \tag{1.4.1}$$

where η is a phase factor and S_i are the rotation operators given by $\frac{\hbar}{2}\sigma_i$ where σ_i are the Pauli matrices.
Sakurai continues by noticing that

$$e^{-\frac{i}{\hbar}S_z\alpha}e^{-\frac{i}{\hbar}S_y(\beta+\pi)}\,|+\rangle = |\hat{n},-\rangle \ , \tag{1.4.2}$$

which can be visualised by thinking of a representation of these vectors on a Bloch Sphere.
Θ is an antiunitary operator, as can be seen from the Schrödinger equation. Thus it can be written in terms of a unitary operator U and the complex conjugate operator K as $\Theta = UK$.
We find

$$\Theta = \eta e^{-\frac{i}{\hbar}\pi S_y} = -i\eta\frac{2S_y}{\hbar}K \tag{1.4.3}$$

and see that this indeed corresponds to a spin-flip operation

$$e^{-\frac{i}{\hbar}\pi S_y}\,|+\rangle = |-\rangle\,; \quad e^{-\frac{i}{\hbar}\pi S_y}\,|-\rangle = -\,|+\rangle \ . \tag{1.4.4}$$

1.5 Quantum Entanglement

Having introduced the necessary preliminaries, this subsection will briefly look at what is arguably one of the most interesting and counter-intuitive aspects of Quantum Theory.
In the discussion of Quantum Entanglement, the concept of separability of a Quantum State is fundamental. What is meant by a state being separable is

a state which can be written in terms of tensor products of single qubit states $|\phi\rangle = |\varphi\rangle \otimes |\chi\rangle$.
As an example, the state

$$|\phi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{1.5.1}$$

is clearly separable, whereas the state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{1.5.2}$$

is non-separable. In fact, it is a *maximally* entangled state, often referred to as a *Bell State*. What's special about this state is that, if one were to measure one of the qubits in the Pauli Z basis, the result obtained would lock-in the result of a similar measurement on the other qubit by collapsing its wavefunction. This process happens instantaneously, even if the two qubits are spatially separated. Importantly, though, this phenomenon does not allow for the superluminal transmission of information. We say that it is *non-signalling*. In this non-relativistic theory of Quantum Mechanics, the laws of Special Relativity are still observed. This is a remarkable fact.
The discussion of entanglement in larger systems, and especially the notion of what it means for a state to be *maximally* entangled, i.e. classifying the *amount* of entanglement present in a Quantum state, will be one of the main points motivating the discussions in subsequent sections.
We will frequently be regarding Quantum Entanglement as a resource that can be used to achieve things that would not otherwise be achievable, such as *Quantum Teleportation*, or to gain an advantage in a game that would not classically be possible to gain. The latter will be discussed at length in the present text.
In the context of entanglement as a resource, one often considers a system consisting of multiple qubits that are spatially separated, and operations that are being performed on the individual qubits. It is common to then consider the individual qubits as being held by different experimenters, commonly named Alice, Bob, Charlie and so forth, holding the A, B and C qubits respectively.
When two parties share an entangled state that is not maximally entangled, potentially due to noise or other imperfections having been introduced, they can perform one of a group of procedures known as *Entanglement Distillation* to probabilistically restore a maximally entangled state from their (ensemble of) less than maximally entangled state(s). Specifically, if the two parties know which (non-maximally entangled) state they are sharing, including the coefficients to the individual terms, and are allowed classical communication, they can distil their single (non-maximally entangled) state probabilistically into a maximally entangled state. This procedure is known as *Procrustean Distillation*, and stands in contrast to *Parity Distillation*, where the shared state can be unknown and

no communication is necessary, but at least two copies of the initial state are required for the procedure to succeed, with the probability of success increasing when a larger number of copies of the state is available. [3]
Thus, probabilistically, a *normalised, physical* two-qubit state that is not maximally entangled provides the same resources as a state that is maximally entangled. This will play an important role when discussing how to quantify entanglement, and will be addressed in more detail throughout the present text.
The discussion above about entanglement distillation is generally also valid for larger than two-qubit systems, however, as one considers larger systems, the mathematical details of the necessary procedures become less clearly defined.

2 Non-Local Games

Quantum Entanglement enables some remarkable, and counter-intuitive to the classical mindset, effects to be observed. To illustrate one example of this behaviour, so-called *Non-Local Games* are discussed in this section. We will introduce some basic two and three-qubit Non-Local Games, or Psychics' Challenges.
The idea of Non-Local Games is to construct games involving players and a referee, where the players are spatially separated such that they cannot communicate after the game commences. The players are then asked questions by the referee, who decides based on their answers whether the players have won or lost. The players are allowed to share a Quantum State, and can agree on a strategy for each game before its commencement.
The games are *non-local* as they are set-up in such a way that the players can exploit non-local properties of their shared Quantum State to perform better than if they weren't sharing the state. An appropriately designed game can thus be used to make the non-local properties of entangled Quantum States apparent, and is intricately related to the violation of Bell type inequalities by such states.
The term Psychichs' Challenge is borrowed from [3], to stress the fact that the results achievable in a non-local game using an appropriate entangled state are not classically reproducible (unless the players were psychic). In addition, the term challenge can be used so as to avoid a possible confusion when using the term game. The games one usually means in this context are cooperative games, in which the players conspire to win a challenge set by the referee, rather than conspiring (and/or cheating) to win against each other in a competitive game, as the terms game or game theory may suggest.

2.1 Two Player CHSH Game

In this section the two player CHSH (Clauser-Horne-Shimony-Holt) Game will be introduced. In order to do so, some necessary concepts, such as the EPR Paradox, Bell and CHSH inequalities are briefly discussed.

2.1.1 Preliminaries, The EPR(B) Paradox

In order to understand what is meant by the EPR(B) paradox, it is helpful to consider the state

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \ . \tag{2.1.1}$$

An important feature of this state is that, if both Alice and Bob measure the spin of their respective qubit in the same direction, they will obtain opposite outcomes. What this means is that, given the spin of one qubit in a certain direction, the spin of the other qubit in that same direction becomes predictable with certainty. This predictability is precisely the requirement for an "Element of Reality" as defined by Einstein, Podolsky and Rosen (EPR) [8]. As this element of reality is not represented in the Quantum Mechanical Theory, EPR concluded that Quantum Mechanics is not a complete theory, and that there must be "hidden variables" (in a classical theory) in which this information is encoded. EPR require such an element of reality to explain the fact that the direction of the spin of one of the qubits could be determined just before the other qubit was measured, such that no information can propagate between the two measurement events. David Bohm [9] later added that a similar paradox arises from the uncertainty when measuring other spin directions, leading to the combined term EPRB Paradox. A Bell type inequality puts a limit on the correlations that can be achieved classically with any local hidden-variable theory, such that a violation thereof implies that the requirements of *local realism* and *completeness* cannot hold.

2.1.2 CHSH Inequality

We will discuss here the CHSH inequality from which the CHSH Game takes its name, a generalisation of Bell's original inequality, as introduced by Clauser, Horne, Shimony and Holt [10]. A Bell type inequality, so named as it was first written down by John Bell [11], is an inequality that puts a classical limit on the probabilities of a thought experiment under the assumption of *Local Realism*.

Local Realism

> What is meant by local realism is the requirement of *Locality* and *Realism*. The assumption of *Realism* says that physical properties should exist in nature independently of observation, while the assumption of *Locality* says that a measurement performed by one party does not influence another measurement made by another party elsewhere, even if the measurements are made on qubits that belong to the same state. [12] *Local Realism* will be discussed in more detail at the end of this section, in paragraph 2.1.2.1.

This brief introduction of the CHSH inequality is following the discussion by Nielsen and Chuang in [12].
First, a classical thought experiment is considered. Two players, Alice and Bob, are given a particle each from a set prepared in some way. Each of them performs one of two measurements (the set of two measurements to choose from is not necessarily the same for the two parties), randomly chosen, on her/his respective particle, and obtains one of two possible outcomes (1 or -1) associated with each measurement. Say Alice either measures the physical properties Q or R, while Bob measures some physical properties S or T. The outcome of all of these measurements is then either $+1$ or -1, and they associate the outcome of the measurement with the respective property. Considering all possible combinations of outcomes ± 1 associated with Q,R,S,T, it then always holds that

$$|QS + RS + RT - QT| = 2 \ . \tag{2.1.2}$$

Defining as $p(q, r, s, t)$ the probability that the measured quantities take the specific values $Q = q, \ R = r, \ S = s, \ T = t$, leads, using (2.1.2), to

$$\begin{aligned} \mathbf{E}(QS + RS + RT - QT) &= \sum_{q,r,s,t} p(q, r, s, t)(qs + rs + rt - qt) \\ &\leq \sum_{q,r,s,t} p(q, r, s, t) \times 2 \\ &= 2 \ , \end{aligned} \tag{2.1.3}$$

where $\mathbf{E}$ denotes the mean value.
The Bell inequality (or CHSH inequality, to be more precise), then follows from the fact that

$$\begin{aligned} \sum_{q,r,s,t} p(q, r, s, t)(qs + rs + rt - qt) &= \sum_{q,r,s,t} p(q, r, s, t)qs + \sum_{q,r,s,t} p(q, r, s, t)rs \\ &+ \sum_{q,r,s,t} p(q, r, s, t)rt - \sum_{q,r,s,t} p(q, r, s, t)qt \\ &= \mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT) \ , \end{aligned} \tag{2.1.4}$$

from which it is seen that

$$\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT) \leq 2 \ . \tag{2.1.5}$$

This is the CHSH inequality, a Bell type inequality. The particles used in this thought experiment, and the measurements performed by the participants, were described in very general terms. It should thus apply to any classical theory, featuring any number of variables, hidden or otherwise. The idea is now to show

that in a Quantum Theory this inequality can be violated, thus showing that the features of the Quantum Theory cannot be accounted for by any classical analogue, and invalidating EPR's claim that the Quantum Effects ought to be accounted for by some hidden variable theory.
For this, now Quantum, thought experiment, two players, Alice and Bob, are considered. They are each being given a qubit from a pair which was prepared in the Bell state (2.1.1), on which they then act with local unitary operations in a specific way, and measure their qubit in the Pauli Z basis. This happens in a causally disconnected fashion, meaning that there is no means for Alice and Bob to communicate and that the operations performed by the players take place in such a way that no information can propagate between them (assuming the laws of Special Relativity).
The operations that can be performed by Alice, given by Q and R, and Bob, given by S and T, are

$$\begin{aligned} Q &= Z \qquad S = \frac{1}{\sqrt{2}}(-Z - X) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \\ R &= X \qquad T = \frac{1}{\sqrt{2}}(Z - X) \quad = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} . \end{aligned} \tag{2.1.6}$$

Q,R,S and T are Quantum Mechanical Observables, and one can compute their expectation value. The violation of the inequality then arises when one considers those expectation values as corresponding to the classical values in (2.1.5), which leads to

$$\langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle = 2\sqrt{2} . \tag{2.1.7}$$

The violation of Bell inequalities has been experimentally verified, notably in [13], and many times since. From this, it can be deduced that the assumption of *local realism* made above cannot hold in a Quantum Theory of nature.

2.1.2.1 More on Local Realism

Having shown that the assumption of local realism is to be rejected in Quantum Theory by Bell's Theorem, a question that arises is whether it is *Locality* or *Realism* that should be abandoned. Agreeing with the summary in [14], arguably, they both should be. Statements that can be interpreted as identifying Realism as contradictory to Quantum predictions are made by the Kochen-Specker Theorem [15] and, more recently and more generally, by the PBR-Theorem [16]. The non-local properties of Quantum Entanglement, which we will meet in due course, suggest that *Locality* cannot hold.

2.1.3 The CHSH Game

The fact that Quantum Mechanics, under some circumstances, violates the CHSH inequality, can be used in different physical settings. In this section we introduce a game commonly used to illustrate the power of entangled Quantum States which are capable of violating Bell type inequalities. In this game the two players will share an entangled Quantum State, and use it to outperform their classical counterparts.
Let the players, Alice and Bob, share a Bell state between them, for example

$$|\Phi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle\right) \ . \tag{2.1.8}$$

A referee sends the players each one classical bit as part of the question, let them be denoted r and s, chosen uniformly at random from the set

$$Q = \{00, 01, 10, 11\} \ . \tag{2.1.9}$$

The players win whenever their answers, denote them by a and b, fulfil the property

$$r \vee s = a \oplus b \ , \tag{2.1.10}$$

where $\vee$ denotes the disjunction and $\oplus$ denotes the sum modulo 2.
Classically, this game can be won by the players at most $\frac{3}{4}$ of the time. A classical strategy that achieves this outcome, for example, would be for the players to always give opposite answers, irrespective of the question.

A Quantum Strategy, making use of the shared Bell pair, that the players can use to perform better than they did classically is laid out in the following.
If Alice is sent a 0 by the referee she performs a measurement on her part of the system in the computational (the Pauli Z) basis, and reports the outcome as her answer.
If Alice is sent a 1 by the referee she performs a measurement of her qubit in the Pauli X (or Hadamard) basis. If she obtains the $|+\rangle$ outcome she responds 0, if she obtains the $|-\rangle$ outcome she responds 1. This is equivalent to her applying a rotation (using a Hadamard Gate) on her qubit first, and then measuring it in the computational basis.
Recall that a generic pair of orthogonal states in the Bloch sphere can e.g. be written

$$|\Psi\rangle = \cos\left(\frac{\alpha}{2}\right)|0\rangle + \sin\left(\frac{\alpha}{2}\right)|1\rangle \quad ; \quad \left|\bar{\Psi}\right\rangle = \sin\left(\frac{\alpha}{2}\right)|0\rangle - \cos\left(\frac{\alpha}{2}\right)|1\rangle \ , \tag{2.1.11}$$

where the angle α is chosen by Bob depending on the question he received from the referee.

If Bob receives a 0 from the referee he measures half-way between the x and z direction in the Bloch sphere, which corresponds to an angle of $\frac{\pi}{4}$. If he receives a 1 from the referee he measures half-way between the negative x and positive z direction, which corresponds to an angle of $\frac{3\pi}{4}$. When Bob obtains $|\Psi\rangle$ he answers 0, when he obtains $\left|\bar{\Psi}\right\rangle$ he answers 1.
It is most easily seen from the Bloch sphere that this is the ideal strategy for Bob to pursue.
Using this strategy we can see that the players will win the game with a probability of $\cos^2\left(\frac{\pi}{8}\right) \approx 0.8536$, thus beating the classical strategy.
As an example, consider the case $rs = 00$. Alice measures in the computational basis, and with a probability of $\frac{1}{2}$ each obtains either the $|0\rangle$ or $|1\rangle$ outcome, and collapses Bob's state to the same. Bob then measures half way between the x and z direction, and they win the game if both Alice and Bob give the same answer (i.e. $0 \oplus 0$ or $1 \oplus 1$). If Alice obtained the $|0\rangle$ outcome, Bob obtains $|\Psi\rangle$, which is the winning outcome, as Bob will answer 0 on this outcome as Alice did, with a probability of $cos^2\left(\frac{\pi}{8}\right)$, as required. Similarly for the other outcomes.

2.2 Three-Player Games

If one considers three parties in a similar game as above, more interesting facets of Quantum Entanglement come to light. In this case the notion of maximal entanglement needs further specification and definition. In the two qubit case we used a Bell state to play the cooperative Quantum Game. Bell states are states commonly accepted as being *maximally* entangled.
However, in the three-qubit case, this notion is not as obvious, as will be discussed in section 4.1. There are, in fact, two ways in which three-qubit states can be considered maximally entangled.
Representatives of the two states which can be considered maximally, genuinely tripartite entangled are

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \ , \tag{2.2.1}$$

the so-called GHZ (after Greenberger, Horne and Zeilinger) or sometimes cat-state, and

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \ , \tag{2.2.2}$$

the so-called W-state. These two types of states feature very different entanglement properties, as we will learn in due course when we play a Quantum Game with them. In short, the GHZ-state (2.2.1) features genuine three-way entanglement between all qubits. The W-state (2.2.2), on the other hand, instead features maximal two-way entanglement between all constituent two-qubit pairs. In fact,

upon tracing out one of the qubits from the GHZ-state the resulting state is unentangled, while tracing out one of the qubits from a W-state leads to a mixed state containing maximal bipartite entanglement. [17]
The resulting mixed state will be a mixture of a Bell state and a separable component. We will see this again in section 4.1.
Given the existence of these inequivalently entangled states, as will be shown below, a game and matching strategy can be devised that lets the players win with certainty when sharing a GHZ state. However, when the state they share is a W-state, they will not be equally successful.

2.2.1 GHZ Game

In this section we will lay out a game and strategy which can be won classically only 75% of the time, but can be won all of the time by the players if they share a GHZ-state. This game appears in many places in the literature, see e.g. [18] and references therein.
As we are considering a game involving three players, the set of questions the referee can ask is given by

$$Q = \{000, 001, 010, 100, 011, 101, 110, 111\} \ . \tag{2.2.3}$$

As in the previous case of the CHSH game, we again require that the players win the game if the condition

$$r \vee s \vee t = a \oplus b \oplus c \ , \tag{2.2.4}$$

is satisfied, where r, s, t are the questions sent to Alice, Bob and Charlie, and a, b, c are their respective answers, $\oplus$ denotes addition modulo 2 and $\vee$ denotes the disjunction.
We limit the set of questions rst the referee will send to

$$\tilde{Q} = \{000, 011, 101, 110\} \ . \tag{2.2.5}$$

In order to evaluate how well the players can do classically in this game, we denote their answers as functions of the respective questions, such that $a = a(r), b = b(s), c = c(t)$. For the question set in (2.2.5) the conditions that the players are successful are then given by the set of equations

$$\begin{aligned} a(0) \oplus b(0) \oplus c(0) &= 0, \\ a(0) \oplus b(1) \oplus c(1) &= 1, \\ a(1) \oplus b(0) \oplus c(1) &= 1, \\ a(1) \oplus b(1) \oplus c(0) &= 1 \ . \end{aligned} \tag{2.2.6}$$

Adding all of these equations together modulo 2 leads to

$$2\Big(a(0) \oplus b(0) \oplus c(0) \oplus a(1) \oplus b(1) \oplus c(1)\Big) = 1, \tag{2.2.7}$$

which cannot be satisfied. This shows that we cannot classically win this game with certainty, and we can convince ourselves that the best classical players can do in this game is win $\frac{3}{4}$ of the time.
In the quantum version the players share a GHZ-state (2.2.1), which it is however convenient to rewrite as

$$\begin{aligned} |GHZ'\rangle = H \otimes H \otimes H \, |GHZ\rangle = \frac{1}{2} \left(|000\rangle + |011\rangle + |101\rangle + |110\rangle \right) \\ \rightarrow |GHZ''\rangle = P \otimes P \otimes P \, |GHZ'\rangle = \frac{1}{2} \left(|000\rangle - |011\rangle - |101\rangle - |110\rangle \right) \ . \end{aligned} \tag{2.2.8}$$

Here P denotes the phase-gate, which is given by $P = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{2}} \end{pmatrix}$.
The strategy for the players to always win this game is as follows. When any of the players receives 0 from the referee she measures in the computational basis and reports the outcome back as her answer. In the case of the referee sending a 1 to the player she measures in the Pauli X basis, which is equivalent to performing a Hadamard rotation on her respective qubit and then measuring it in the Pauli Z basis. The outcome will again be reported as the answer.
As an example, in the case that the question sent by the referee is $rst = 000$, all players measure their respective qubit of (2.2.8) in the Z basis, and we can see that they always obtain an even number of 1 outcomes, and the requirement (2.2.4) is always satisfied.
For the other three possible questions in the set (2.2.5) it is enough to consider one question, as they are all related by a permutation symmetry, which, importantly, is also true for the state shared between the parties (2.2.8).
Considering the case $rst = 011$, this corresponds to the players performing

$$\mathbb{1} \otimes H \otimes H \, |GHZ''\rangle = \frac{1}{2} \left(|001\rangle + |010\rangle - |100\rangle + |111\rangle \right) \tag{2.2.9}$$

on the state and subsequently each measuring their qubit in the Z basis and reporting the outcome as their answer.
We can immediately see that they will always report back an odd number of 1 outcomes, such that they are successful.
Thus, in this scenario, the players sharing a GHZ-state will win 100% of the time. An easy to understand Gedankenexperiment to illustrate this behaviour is given by Mermin in [19].

2.2.1.1 W Performance in The GHZ Game

As was very briefly pointed out at the beginning of this section, both the GHZ and the W-state feature some notion of *maximal* entanglement. But we also noted that these states are entangled in very different ways. So the question that naturally arises is whether or not the players in above GHZ Game are equally at an advantage if the state they share is of the W-type instead of the GHZ-type. In order to answer this question, consider the state

$$|W'\rangle = \frac{1}{\sqrt{3}} \left(|011\rangle + |101\rangle + |110\rangle \right) = X \otimes X \otimes X \, |W\rangle \ , \tag{2.2.10}$$

which is locally equivalent to the W-state, as shown. If our players share this state and play the same game as above, looking at the two different scenarios $rst = 000, 011$ is again sufficient due to the permutation symmetry in the state, as well as the rest of the question set.
If the referee sends $rst = 000$ to the players, none of them perform any local operations on their qubit and measure it in the Pauli Z basis, leading to an outcome that always features an even number of 1's, and thus they always win when this question is being asked.
If the referee sends $rst = 011$ to the players, they perform the operation

$$\mathbb{1} \otimes H \otimes H \, |W\rangle = \frac{1}{2\sqrt{3}} \left(|000\rangle + |011\rangle - |001\rangle - |010\rangle + 2\,|100\rangle - 2\,|111\rangle \right) \ , \tag{2.2.11}$$

and, in this case, will win if their answers satisfy $a \oplus b \oplus c = 1$, which means that they lose for the outcomes $|000\rangle$ and $|011\rangle$, which are obtained with a probability of $\frac{1}{6}$. Thus, the winning probability in this case is $\frac{5}{6}$, and the same is true for the questions $\tilde{Q} = 101, 110$. The total probability for the players sharing a W-type state to win this game is then $p_{win} = \frac{1}{4}\left(1 + \frac{5}{6} + \frac{5}{6} + \frac{5}{6}\right) = \frac{7}{8}$.
We see that the players cannot achieve the same performance they did sharing a GHZ-state while sharing a W-state. It can be shown that there is *no* strategy, using the the W-state, which wins this game with certainty [20]. This makes the difference in entanglement between GHZ and W-type states manifest. We will encounter a situation in which the W-state outperforms the GHZ-state in section 2.2.2.

2.2.1.2 Bi-Separable Performance in GHZ Game

Another state to consider in the above scenario is a state which features only two-qubit entanglement between two constituent qubits, and hence is not maximally entangled. Such a state is known as *Bi-Separable*, and will highlight once more the need for an accurate classification of the *amount* and kind of entanglement contained in a state, which is the subject of section 4.1.
If we play the exact same GHZ Game with the players sharing a Bi-Separable State, say

$$|A-BC\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle - |011\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle\left(|00\rangle - |11\rangle\right)\ , \tag{2.2.12}$$

in which the A-qubit is completely separable, while the B and C qubit are maximally entangled (being part of a Bell pair), it can be seen that employing the same strategy will not lead to an advantage over the classical case. In particular, we can see that for the questions $\tilde{Q} = \{000, 011\}$ the players can win all of the time, whilst for the questions $\tilde{Q}' = \{101, 110\}$ they only win half of the time, such that the overall probability of winning the game for these players is $\frac{3}{4}$, as it was in the classical case. However, notice that, since the A-qubit is separable, upon ignoring the question the referee sends to Alice (say Alice always answers 0 regardless of what question she received), the question set reduces to $Q = \{00, 01, 10, 11\}$, which is precisely the set of questions encountered in the CHSH Game in section 2.1.3.
Thus, if the players in this case change their strategy to that of the CHSH Game, they can also achieve the same winning probability of 0.8536. Looking, for example, at the case $rst = 110$, following the strategy from section 2.1.3, the players apply local operations to the state such that

$$\begin{aligned}|A-BC\rangle' = \frac{1}{2}\Big[&|0\rangle\left(|0\rangle + |1\rangle\right)\left(\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle\right)\\ &+ |0\rangle\left(|0\rangle - |1\rangle\right)\left(-\sin\frac{\pi}{8}|0\rangle + \cos\frac{\pi}{8}|1\rangle\right)\Big]\ .\end{aligned} \tag{2.2.13}$$

The winning probability is thus $\frac{1}{2}\left(\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right)^2 \approx 0.8536$, as in the CHSH case. Similarly for the other possible questions.

2.2.2 Advantage W

In the previous section it was shown that three players sharing a GHZ-type state are able to win a special type of non-local game 100% of the time that classically could only be won 75% of the time. However, sharing a W-type state in the same game did not result in the same advantage. In order to let the W-state be advantageous over the GHZ-state, the game needs to be modified.
As was pointed out, the W-state features maximal entanglement between the individual two-qubit pairs, whereas the GHZ-state features genuine three-way entanglement, and in fact becomes separable upon the tracing out one of its constituent qubits.
This suggests that a game in which the referee starts with the same question set as in the GHZ-game, $\tilde{Q} = \{000, 011, 101, 110\}$, but randomly chooses to ignore one of the players, thus really playing a CHSH-game with two of the three players randomly chosen, will be won by players sharing a W-type state with the same probability that a CHSH Game can be won, namely 85.36%. The players simply

ignore the player to which no question is sent, effectively playing a two player game. This amounts to tracing out the player that (as randomly chosen by the referee) does not participate, leaving the two players that do with a mixed, but maximally entangled state between them. Noting again that, upon tracing out one of the players the GHZ-state offers no residual entanglement, players sharing a GHZ-type state in the same scenario will have no advantage over the classical case.

The analysis above relies on two of the players who initially shared a three-qubit W-state to share a maximally entangled Bell pair after one qubit is traced out. However, we saw previously that the state shared by the parties after tracing out one qubit is, in fact, mixed. So to justify the assumption of the players sharing a pure Bell pair, a very important observation needs to be made. The mixed two-qubit state obtained when tracing out one qubit from the three-qubit W-state can be transformed into a state arbitrarily close to a pure Bell state using local filtering operations. [17, 21] One can think of this as akin to the distillation procedures discussed previously. Although this may be a probabilistic process, that does not fundamentally change the result of this game. Depending on protocol, it may be possible for the success rate of the filtering operation to be increased arbitrarily close to unity by allowing the parties to share an arbitrary number of copies of the state. This would not improve the chances of winning when the players share a GHZ-state. But even assuming that the filtering operation will always be probabilistic and cannot be improved by sharing multiple copies of the state, after a sufficiently large number of rounds of the game being played, sharing a W-state would still be advantageous. The probability of winning would in this case be less than the 85.36% from above, but would still exceed the classical 75%, which is also the maximum players sharing a GHZ-state could achieve.

In the game discussed here a Bi-Separable State will offer an advantage over the classical case in $\frac{1}{4}$ of the runs of the game, whenever the referee chooses to not send a question to the player whose qubit is separable in the shared state. In all other cases, holding the Bi-Separable State is not advantageous to the players, leading to a probability of winning of $p_{win} \approx 0.7759$.

Although we have seen that the W-state will outperform the GHZ-state in this type of game, it is a rather contrived game testing residual two-qubit entanglement, rather than testing genuine tripartite entanglement. A game in which the W-state has an advantage over the GHZ-state that involves all three players rather than choosing a subset would be very desirable to have, as it may enable one to make a more clear statement as to the differentiation of the two notions of maximal entanglement present in these two types of states. Similar is true for a game in which the W-state gains a 100% advantage and the GHZ-state does not.

3 Quantifying Quantum Entanglement

In the previous section on Quantum Games we saw that there are two three-qubit states, completely non-separable, which can both be considered maximally entangled, and yet feature some very different properties. The GHZ-state and the W-state. As one considers even larger systems, one encounters even larger numbers of non-separable states with vastly different properties. This underlines the need for a more fine-grained classification of entangled states beyond separability criteria.

In order to facilitate these discussions, one needs to introduce a notion of equivalence between different Quantum States based on their entanglement properties, which is the subject of this section.

3.1 LOCC

The term *LOCC* stands for *Local Operations* and *Classical Communication*, and represents the combination of local operations performed in combination with a classical means of communication between the parties involved. In the study of Quantum Entanglement *LOCC* operations are important, as they define *classical* correlations as those which can be created using *LOCC* operations. [22] To make the connection to Quantum Mechanical Correlations, it is important to define the notion of *Local Operations*, which is also necessary in order to define a consistent notion of equivalence between states.

A natural way of defining *Local Operations* is by allowing only operations given by *Local Unitary Operators*. [23] These Operators can notably not create or destroy Quantum Mechanical Correlations. In order to account for all possible classical correlations under the paradigm of *LOCC*, in addition to *Local Unitary Operators*, one includes *Local General Measurements*, which also enables the destruction of Quantum Mechanical Correlations, but not their creation. [24] As an example, one can think of two players sharing a Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, where upon, say, Alice measuring her qubit in the Pauli Z basis, depending on outcome the state collapses to $|00\rangle$ or $|11\rangle$, both of which have become separable.

In summary, we include in the class of *LOCC* Operations Local Operations together with Classical Communication. *Local Operations* include all possible operations that can be performed by a party on their individual system, including [25, 26]

- Local Unitary Operations
- Adding Local Systems (Ancillae)
- Deleting (tracing out) Local Systems
- Local Measurements

and in *LOCC* Classical Communication is allowed in all directions. Note once more that these operations allow for the destruction of Quantum Entanglement, but cannot create Quantum Entanglement between the constituent qubits (locally Quantum Entanglement could be created between e.g. an Ancilla Qubit introduced by one party and their original qubit).

3.2 LOCC Equivalence

We noted above that *LOCC* Operations can destroy, but not create any new Quantum Entanglement between the constituent qubits of the state. A state that has been acted on with an LOCC Operation is thus equally or less entangled than the original state. It follows that two states that can be interrelated deterministically using LOCC Operations will necessarily contain the same amount of entanglement. This observation is what we use to define *LOCC Equivalence*. Two states are LOCC *equivalent* if they can be interrelated with certainty, using LOCC Operations. LOCC equivalent states feature the same amount of entanglement.

Defining any valid Quantum Operation as a Superoperator $\mathcal{E}$ acting on the Quantum System (given in terms of a density matrix ρ), it follows from the discussion on POVMs in section 1.3 that the probability of this transformation occurring is given by $\operatorname{Tr}\mathcal{E}(\rho)$.

The state will then transform as

$$\rho \to \frac{\mathcal{E}(\rho)}{\operatorname{Tr}(\mathcal{E}(\rho))} \ , \tag{3.2.1}$$

so as to make sure the result is still a valid density operator with unit trace.

Recall that *LOCC* equivalence was defined as a deterministic interrelation between states, such that $\operatorname{Tr}(\mathcal{E}(\rho)) = 1$. [27]

This is a more restrictive set than that of all *LOCC* Operations as defined above, and in short one can define equivalent states under *LOCC* as those which can be reversibly interrelated with certainty, i.e. using *Local Unitary Operators* aided by *Classical Communication* [23].

The group of *Local Unitaries* (now considering a general system of n qudits) is $[U(d)]^{\times n}$, which, upon noting that this includes the multiplication by an overall factor, reduces to $U(1) \times [SU(d)]^{\times n}$. [28]

We can use this to define orbits of states equivalent under *Local Unitaries* as

$$\frac{\mathbb{C}^{d^{\otimes n}}}{U(1) \times [SU(d)]^{\times n}} \ , \tag{3.2.2}$$

which, as explained in section 1.2, if we only care about physical states, simplifies to

$$\frac{\mathbb{CP}^{d^n-1}}{[SU(d)]^{\times n}} \ , \tag{3.2.3}$$

where in both equations for a qubit we have $d = 2$.
However, recall from the discussion on Entanglement in section 1.5 that, when considering entanglement as a resource, we considered a state or ensemble of states which could be *distilled* to be maximally entangled as equivalent to a maximally entangled state. The distillation procedures were probabilistic, so it becomes obvious that limiting ourselves to deterministic *LOCC* equivalence when discussing the amount of entanglement a state contains is too restrictive.

3.3 SLOCC Equivalence

In order to define a less restrictive notion of equivalence of states in terms of the amount of entanglement they contain, it is thus useful to include operations which can only stochastically interrelate quantum states. States are then defined as *SLOCC* equivalent similarly to LOCC equivalence defined earlier, only now including operations that will only stochastically (with a probability $0 < p \leq 1$) interrelate states.
This criterion will include distillation procedures, such as the ones introduced previously, and find states to be equivalent if the same operations, which use entanglement as a resource, can be performed with them with some non-zero probability.
The name *Stochastic Local Operations and Classical Communication (SLOCC)* has been established for this paradigm. Formally, easing the restriction on the success rate of the operations to allow it to be less than unity is represented by letting the map $\mathcal{E}$ be not necessarily trace preserving, such that [27]

$$\mathrm{Tr}\left(\mathcal{E}(\rho)\right) \leq 1 \ . \tag{3.3.1}$$

SLOCC equivalence is a coarse graining of the *LOCC* equivalence defined in section 3.1, in that states equivalent under *LOCC* are naturally also equivalent under *SLOCC*, and we have that two states can be regarded as equivalent under *SLOCC* operations if they are related by *Invertible Local Operators*. [29]
In analogy to the discussion of *LOCC* equivalence in section 3.2, the group of *Invertible Local Operations* making up the *SLOCC* equivalence group (for n qudits) is $[SL(d,\mathbb{C})]^{\times n}$ and includes the multiplication by an overall complex factor, such that the orbit of *SLOCC* equivalent states becomes [17]

$$\frac{\mathbb{C}^{d^{\otimes n}}}{[SL(d,\mathbb{C})]^{\times n}} \ . \tag{3.3.2}$$

Note again that we have considered the Hilbert Space of all states rather than the complex projective space, as $SL(d, \mathbb{C})$ does not preserve the norm. The case for a qubit is again recovered for $d = 2$.
In order to classify the different ways in which Quantum States of a certain number of qubits can be entangled, one approach is to classify the different orbits in equation (3.3.2) using quantities that are invariant or covariant under the SLOCC group action, and observing whether or not they vanish for a given orbit. We will make use of this approach in section 5.

3.4 Entanglement Measures

Having defined the notion of equivalently entangled states lets us define mathematical quantities to measure the *amount* of entanglement contained in two and three-qubit quantum states. For the case of mixed three qubit states evaluating these quantities becomes an involved numerical issue, and is not completely settled. These measures can be used to distinguish entangled states using the notion of *SLOCC equivalence*, but are not manifestly *SLOCC* covariant, which will become important later on in the discussion.

There are several requirements a quantity should fulfil in order to be considered a valid entanglement measure.

- The measure is a map from the state to a real number. [22]
- The measure should be *monotonically* decreasing under *LOCC* Operations [30, 24], as we know that these can destroy entanglement by the extended definition above.
- Consequently, the measure should be invariant under *ILO*, as they are a subset of the *LOCC* Operations.

A measure which satisfies these conditions will generally be referred to as an *Entanglement Monotone*. [22] While it is clear that an entanglement monotone must be invariant under *ILO*, perhaps more surprisingly the converse is also true. Any (linearly homogeneous) *ILO* invariant function of a pure (unnormalised) state is also an entanglement monotone. [31]
In the following we will discuss some entanglement measures that can be used in different contexts and are important for the discussions to follow.

3.4.1 Positivity of Partial Transpose

The Positivity of the Partial Transpose (also the Peres, or Peres-Horodecki criterion, see [32] and [33]) is a necessary criterion for a (potentially mixed) *bipartite*

quantum state to be separable, and in some cases, namely the two-qubit and qubit-qutrit case, is also sufficient.
A *Positive* Map Φ acting on one qubit is defined such that [6]

$$\Phi\left(\rho_A\right) \geq 0 \qquad \forall \rho_A \ , \tag{3.4.1}$$

and a *Completely Positive* (CP) Map such that

$$\mathbb{1}_A \otimes \Phi_B\left(\rho_{AB}\right) \geq 0 \qquad \forall \rho_{AB} \ . \tag{3.4.2}$$

A *Not Completely Positive* (NCP) Map is then defined such that the Map is positive, implying that equation (3.4.1) holds, but not completely positive, implying that equation (3.4.2) does not hold.
The Partial Transpose is a positive, but not completely positive map. It holds that any NCP map Φ preserves the positivity of a quantum state if it is separable when one considers the partial action [33, 6]

$$\mathbb{1}_A \otimes \Phi_B\left(\rho_{AB}^{sep}\right) \geq 0 \qquad \forall \rho_{AB}^{sep} \ . \tag{3.4.3}$$

Thus this can be used as a necessary criterion for separability.
The action of the Partial Transposition is defined as the Transposition on one of the subsystems of the state. For the two qubit case in which this criterion is particularly useful, one may look at the Partial Transpose with respect to e.g. the first subsystem

$$\left|11\right\rangle \left\langle 00\right| \rightarrow \left|01\right\rangle \left\langle 10\right| \ . \tag{3.4.4}$$

If the density matrix after the Partial Transposition (where it is not important with respect to which subsystem the Partial Transposition is carried out) is positive (i.e. has no negative eigenvalues), the state is separable. For two-qubit and qubit-qutrit systems it also holds that, if the system after Partial Transposition is not positive, it is entangled.

3.4.2 Von Neumann Entropy

The von Neumann Entropy is defined as [34]

$$S(\rho) = -\operatorname{Tr}\left(\rho \log_2(\rho)\right) \ , \tag{3.4.5}$$

or in terms of eigenvalues λ_i of ρ

$$S(\rho) = -\sum_i \lambda_i \log_2\left(\lambda_i\right) \ . \tag{3.4.6}$$

The von Neumann Entropy can be used as a measure of the purity of a (single qubit) Quantum State, with a pure state having $S = 0$ and a maximally mixed state having $S = 1$. [34, 6]

When considering a pure two qubit state, it is entangled iff the state of either of the two subsystems (after tracing out the other subsystem) is mixed. This is proven by proving the converse, that if the state of the subsystem is pure, the original two qubit state is separable, following [6].
For a separable, pure two qubit state, we have that $|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, and hence $\rho_A = |\Psi_A\rangle\langle\Psi_A|$.
In terms of a basis $|\Psi_{AB}\rangle = \sum_{i,k} \Psi_{ik}|ik\rangle$, from which it follows that

$$\rho_A = \sum_{i,k,l,i',k'} \Psi_{ik}\Psi^*_{i'k'}\ {}_B\langle l\,|ik\,\rangle\langle i'k'|\,l\rangle_B = \sum_{i,k} |\Psi_{ik}|^2\,|i\,\rangle\langle i'| \quad . \tag{3.4.7}$$

Then, defining $c_i \equiv \sum_k |\Psi_{ik}|^2$, leads, when imposing purity of the state $\rho_A^2 = \rho_A$, to $i = i'$ and $c_i^2 = c_i$, from which it follows that either $c_i = 1$ or $c_i = 0$. Noting the general requirement of a density operator that it has unit trace, it follows that $\sum_i c_i = 1$, and thus $c_1 = 1$, $c_i = 0 \quad \forall i \neq 1$.
It follows then that $\sum_k |\Psi_{1k}|^2 = 1$ and $\sum_{ik} |\Psi_{ik}|^2 = 0 \quad \forall i \neq 1$, which leads to

$$|\Psi_{AB}\rangle = \sum_{ik} \Psi_{ik}\,|ik\rangle = \sum_k \Psi_{1k}\,|1k\rangle \quad , \tag{3.4.8}$$

which is separable, such that the claim is proven.

3.4.3 Local Rank

The *Local Rank* is an important concept in the discussion of the *pure* state entanglement of two qubits. The *rank* of a reduced (local) density matrix $(r(\rho_A), r(\rho_B))$, which coincides with the *Schmidt Number* of that matrix given by n_ψ in the *Schmidt Decomposition*

$$U_A \otimes U_B\,|\psi\rangle = \sum_{i=1}^{n_\psi} \sqrt{\lambda_i}\,|i\rangle \otimes |i\rangle \quad , \tag{3.4.9}$$

is invariant under *SLOCC* Operations, leading to the conclusion that given $\rho \in \mathbb{C}^n \otimes \mathbb{C}^m$, where $n_\psi \leq n \leq m$, there are at most n different ways of entangling the states. [17] Dür et al. note further that the local rank provides a hierarchical entanglement classification since states with $n_\psi = 1$ are separable, and non-invertible *LOCC* Operations can project out terms from the *Schmidt Decomposition* and hence diminish n_ψ as required.
This measure thus bears a close similarity to the local entropy introduced earlier.

3.4.4 Entanglement of Formation

The "Entanglement of Formation" is a measure of entanglement defined in terms of the decomposition of the density operator into state vectors. Presumably, it takes its name from the fact that it is a measure of the entanglement contained in the state vectors that form the density operator, and hence the total Quantum

State. As such, it is also defined for mixed Quantum States. Given a mixed state written in terms of a pure state decomposition

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \ , \tag{3.4.10}$$

for some ensemble $\{p_i, \psi_i\}$, the Entanglement of Formation is defined as the minimum of the average amount of entanglement of each such decomposition of the given mixed state, where the minimum is taken over all possible decompositions [30]. Formally, we have [35]

$$E(\rho) = \min \sum_i p_i E(\Psi_i) \ , \tag{3.4.11}$$

where $E(\Psi_i) = E(\left|\psi_i\right\rangle \left\langle\psi_i\right|)$ is the entanglement of the i-th bipartite pure state in the decomposition (3.4.10), given by the local von Neumann Entropy of either subsystem.
This measure also reduces to the von Neumann Entropy of either subsystem in the case of a bipartite pure state, and to the average thereof if applied to an ensemble of bipartite pure states. [30]
As is shown in the respective references, the Entanglement of Formation is an entanglement monotone, which follows from the fact that the von Neumann Entropy is. Thus, the Entanglement of Formation is an appropriate measure of entanglement for the cases considered.
However, as this measure includes an extremisation problem, one can define other, more convenient measures of entanglement of mixed states.

3.4.5 Concurrence

The goal in defining the Concurrence is to find an explicit expression (for specific cases) for the Entanglement of Formation, so as to avoid having to evaluate the extremisation problem. This is done in [36] and [35], the most important definitions and arguments of which we will lay out in this subsection.
Defining the spin-flipped state $\left|\tilde{\psi}\right\rangle = \sigma_y \left|\psi\right\rangle^*$, consistent with the definition given in section 1.4, the Concurrence is given by

$$C(\psi) = \left|\left\langle \psi \middle| \tilde{\psi} \right\rangle\right| \ . \tag{3.4.12}$$

The amount of entanglement, given by the von Neumann Entropy $E(\psi)$, can then be written in terms of the Concurrence as

$$E(\psi) = \mathcal{E}(C(\psi)) \ , \tag{3.4.13}$$

where the function $\mathcal{E} = -\frac{1+\sqrt{1-C^2}}{2} \log_2 \frac{1+\sqrt{1-C^2}}{2} - \frac{1-\sqrt{1-C^2}}{2} \log_2 \frac{1-\sqrt{1-C^2}}{2}$.
The Concurrence can be generalised to mixed states. To this end, note that the spin-flipped density matrix (for a 2-qubit system) is given by

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \ . \tag{3.4.14}$$

The Concurrence is then given by

$$C(\rho) = \max\{0, (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)\} \ , \tag{3.4.15}$$

where the λ_i are the eigenvalues of the matrix $R \equiv \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ or, equivalently, the square roots of the eigenvalues of the matrix $\tilde{R} \equiv \rho\tilde{\rho}$ in decreasing order with respect to the labels i [36, 35].

3.4.6 Tangles

Having introduced the concept of the Concurrence enables us to treat of another very important concept when quantifying entanglement, the Tangles.
We follow the treatment in [37], adding additional detail and clarification where necessary.
The authors begin by defining the Tangle as the square of the Concurrence, in terms of the eigenvalues of the matrix $\tilde{R}$ as given in (3.4.15), and note that for the case of a pure state of systems A and B, the matrix $\tilde{R}$ only has one non-zero eigenvalue, leading to the expression for the tangle $\tau_{AB} = 4 \det \rho_A$.
In order to generalise to the case of a pure state of three qubits, they note further that in this case, the density matrix for each respective pair of qubits has two non-zero eigenvalues, as each pair of qubits is only entangled with one other qubit. This observation leads to the expression

$$\tau_{AB} = (\lambda_1 - \lambda_2)^2 = \mathrm{Tr}\left(\rho_{AB}\tilde{\rho}_{AB}\right) - 2\lambda_1\lambda_2 \leq \mathrm{Tr}\left(\rho_{AB}\tilde{\rho}_{AB}\right) \ , \tag{3.4.16}$$

which enables one to write

$$\tau_{AB} + \tau_{AC} \leq \mathrm{Tr}\left(\rho_{AB}\tilde{\rho}_{AB}\right) + \mathrm{Tr}\left(\rho_{AC}\tilde{\rho}_{AC}\right) \ . \tag{3.4.17}$$

Then, noting the definition of the spin flip in (3.4.14) and writing $\rho_{ABC} = \sum_{ijk}\sum_{mnp} a_{ijk}a^*_{mnp} |ijk\rangle\langle mnp|$, we find

$$\tilde{\rho}_{ABC} = \varepsilon_{mm'}\varepsilon_{nn'}\varepsilon_{pp'}\varepsilon_{ii'}\varepsilon_{jj'}\varepsilon_{kk'}a_{mnp}a^*_{ijk} |m'n'p'\rangle\langle i'j'k'| \ , \tag{3.4.18}$$

where we have omitted the summation symbol, making it implicit using the Einstein summation convention.
To find the quantity we are interested in, $\rho_{AB}\tilde{\rho}_{AB}$, it is noted that after tracing out the C qubit, one has $\rho_{AB} = a_{ijk}a^*_{mnp}\delta_{kp} |\, ij\rangle\langle\, mn|$, where δ is the Kronecker Delta symbol. Using this result, we have

$$\begin{aligned}\rho_{AB}\tilde{\rho}_{AB} &= \varepsilon_{aa'}\varepsilon_{bb'}\varepsilon_{cc'}\varepsilon_{rr'}\varepsilon_{ss'}\varepsilon_{tt'}a_{ijk}a^*_{mnp}a_{abc}a^*_{rst}\delta_{kp}\delta_{c't'} |ij\,\rangle\langle mn\,|a'b'\,\rangle\langle r's'| \\ &= \varepsilon_{aa'}\varepsilon_{bb'}\varepsilon_{cc'}\varepsilon_{rr'}\varepsilon_{ss'}\varepsilon_{tt'}a_{ijk}a^*_{mnp}a_{abc}a^*_{rst}\delta_{kp}\delta_{c't'}\delta_{ma'}\delta_{nb'} |ij\,\rangle\langle r's'| \ ,\end{aligned} \tag{3.4.19}$$

and noting that $\varepsilon_{ac}\varepsilon_{bc} = \delta_{ab}$, and taking the overall trace, this leads to

$$\mathrm{Tr}\left(\rho_{AB}\tilde{\rho}_{AB}\right) = a_{ijk}a^{*}_{mnk}a_{abt}a^{*}_{rst}\varepsilon_{am}\varepsilon_{bn}\varepsilon_{ri}\varepsilon_{sj} \ . \tag{3.4.20}$$

The authors then use the fact that $\varepsilon_{aa'}\varepsilon_{b'b} = \delta_{ab'}\delta_{a'b} - \delta_{ab}\delta_{a'b'}$ to note that, continuing from the above,

$$\underbrace{a_{ijk}a^{*}_{mnk}a_{rjt}a^{*}_{ant}\varepsilon_{mr}\varepsilon_{ai}}_{(\star)} - \underbrace{a_{ijk}a^{*}_{mjk}a_{ast}a^{*}_{rst}\varepsilon_{mr}\varepsilon_{ai}}_{(\star\star)} \ , \tag{3.4.21}$$

for which we can see, using $\varepsilon_{mr}\varepsilon_{ai} = \delta_{ma}\delta_{ri} - \delta_{mi}\delta_{ar}$

$$\begin{aligned}(\star) &= \underbrace{a_{ijk}a^{*}_{ank}a_{ant}a_{ijt}}_{\left[a_{ij}a^{*}_{ij}\right]_{kt}\left[a_{an}a^{*}_{an}\right]_{tk}} - a_{ijk}a^{*}_{ink}a_{ant}a^{*}_{ajt} \\ &= \mathrm{Tr}\left(\rho_C^2\right) - \mathrm{Tr}\left(\rho_B^2\right) \\ (\star\star) &= -\left(\rho_A\right)_{im}\left(\rho_A\right)_{ar}\varepsilon_{mr}\varepsilon_{ai} \\ &= 2\det\rho_A \ .\end{aligned} \tag{3.4.22}$$

In total, noting that $\det A = \frac{1}{2}\left(\mathrm{Tr}\,A\right)^2 - \mathrm{Tr}\left(A^2\right)$ and $\mathrm{Tr}\,\rho_\kappa = 1$

$$\begin{aligned}\mathrm{Tr}\left(\rho_{AB}\tilde{\rho}_{AB}\right) &= 2\det\rho_A - \mathrm{Tr}\left(\rho_B^2\right) + \mathrm{Tr}\left(\rho_C^2\right) \\ &= 2\det\rho_A + 2\det\rho_B - 2\det\rho_C \ .\end{aligned} \tag{3.4.23}$$

It follows, using (3.4.17), that

$$\tau_{AB}\tau_{AC} \leq 4\det\rho_A \ . \tag{3.4.24}$$

The authors note further that, as ρ_{BC} only has two non-zero eigenvalues, it can be treated as effectively two dimensional when expressing a state of the three qubits ABC. Thus, it is valid to consider the tangle between A and the pair of qubits BC, and one finds $\tau_{A(BC)} = 4\det\rho_A$, such that

$$\tau_{AB} + \tau_{AC} \leq \tau_{A(BC)} \ , \tag{3.4.25}$$

which also makes sense intuitively.
At this point the authors observe that the tangle $\tau_{A(BC)}$ is not defined for mixed states. This is a problem that we will face again when talking about four-qubit entanglement, as the residual three-qubit states after tracing out one qubit can be mixed, and the tangle for three qubits (3-tangle) is also not defined for mixed states.

A generalisation of $\tau_{A(BC)}$ to mixed states can be achieved similarly to the definition of the *entanglement of formation* in section 3.4.4. The idea is to maximise the tangle for each of the possible pure state decompositions of the mixed state, which is the same way the *entanglement of formation* is defined. However, that is a non-trivial problem which may be difficult to solve analytically.
It is further shown by the authors that there exist no other function of τ that provides a stricter bound than that given in equation (3.4.25).
The above discussion can be used to define a tangle that measures the genuine tripartite entanglement of ABC. This quantity will be referred to as the 3-tangle throughout, denoted here by τ_{ABC}. The key idea is to consider the difference between the LHS and RHS in equation (3.4.25) which will account for the entanglement present between A and the BC pair which cannot be accounted for by the individual entanglement of A with B and A with C.
As shown by the authors, the 3-tangle is given by

$$\tau_3 = \tau_{ABC} = 2\left|a_{ijk}a_{i'j'm}a_{npk'}a_{n'p'm'}\varepsilon_{ii'}\varepsilon_{jj'}\varepsilon_{kk'}\varepsilon_{mm'}\varepsilon_{nn'}\varepsilon_{pp'}\right| \ , \qquad (3.4.26)$$

a quantity that is also known (up to pre-factors) as *Cayley's Hyperdeterminant* as noted in [27].
The tangle, as written in equation (3.4.26), is manifestly *SLOCC* covariant, which is not true of the other measures introduced. Apart from the Concurrence, these other measures are invariant under the *LOCC* equivalence relation and hence under the action of *Local Unitaries.*

3.4.7 Four-Qubit Measures

For quadripartite and larger systems, defining unambiguous, simple measures of entanglement remains an open problem. Similarly, it is difficult to analyse the behaviour of four-qubit states when they are reduced to three-qubit states (by tracing out one of the qubits), as the resulting states are potentially mixed, and the 3-tangle is only defined for pure states.
A generalisation of the *Concurrence* and the *3-tangle* to four qubits is constructed in [31].
The authors in [31] conclude that a total of 6 independent entanglement measures can be constructed for the four-qubit case. Among them a generalisation of the *Concurrence* to 4 qubits

$$C_4 = \left|a_{i_1j_1k_1l_1}a_{i_2j_2k_2l_2}\varepsilon_{i_1i_2}\varepsilon_{j_1j_2}\varepsilon_{k_1k_2}\varepsilon_{l_1l_2}\right| \ , \qquad (3.4.27)$$

and a generalisation of the *3-tangle* to 4 qubits

$$\tau_4 = \sqrt{2}\left|a_{i_1k_1j_1l_1}a_{i_2j_2k_2l_2}a_{i_3j_3k_3l_3}a_{i_4j_4k_4l_4}\varepsilon_{i_1i_2}\varepsilon_{i_3i_4}\varepsilon_{l_1l_2}\varepsilon_{l_3l_4}\varepsilon_{j_1j_3}\varepsilon_{j_2j_4}\varepsilon_{k_1k_3}\varepsilon_{k_2k_4}\right|^{\frac{1}{2}} \ , \qquad (3.4.28)$$

where a_{ω_κ} for $\omega = i, j, k, l$, $\kappa = 1, 2, 3, 4$, $\omega_\kappa = 0, 1$ are the coefficients to the respective ket vectors $|\omega_1\omega_2\omega_3\omega_4\rangle$ and ε is the totally anti-symmetric Levi-Civita Tensor.
These measures will be further explored in section 4.2, where the different ways in which four-qubit states can be entangled are discussed. It will be noted then that these two measures are insufficient to classify the entanglement of four qubits, and a sufficient measure that does not amount to evaluating all covariant / invariant quantities is still lacking.

4 SLOCC Entanglement Classification

How many different ways of entangling n-qubits exist in terms of *SLOCC equivalence* will be discussed for the cases of three and four qubit states in the following sections.
However, this classification doesn't easily generalise to larger systems, and hence a more systematic, generalisable approach is desired, and one way of achieving this is the subject of section 5.
For systems smaller than four qubits a formalism to classify the amount of entanglement contained in a state exists that is commonly used and accepted, so long as one confines themselves to the case of pure states. This formalism will be summarised in this section, and some of the problems that occur on attempting to generalise it to four-qubit systems are pointed out. This ultimately results in the pursuit of a different formalism for the classification of entanglement, which is elaborated on in section 5.

4.1 Three-Qubit Entanglement

We briefly summarise here the *SLOCC* classification of *pure* three-qubit entanglement by Dür et al. in [17].
The concepts introduced in section 3.4 will be of importance for this and the following discussions.
Dür et al. found that there are two inequivalent ways of maximally entangling three qubit states (referred to as the W and GHZ classes of states), along with three equivalent ways of bipartite entangling only two of the three qubits. All non-separable three-qubit states are either of the GHZ or W class. In detail, the authors found the following different classes of entangled states of three qubits.

Separable

Product states which are completely separable. For these states it holds that all the local ranks are equal to unity, $r(\rho_A) = r(\rho_B) = r(\rho_C) = 1$, and a representative state is

$$|A - B - C\rangle = |000\rangle \ . \tag{4.1.1}$$

Bipartite Ent.

States which feature entanglement between two of the constituent qubits, with the third being separable. For example, consider the state with bipartite entanglement between the B and C qubits, with the A qubit separable. For this state it holds that $r(\rho_B) = r(\rho_C) = 2$ and $(\rho_A) = 1$. A representative can be written as

$$|A - BC\rangle = |0\rangle \left(c_\phi |00\rangle + s_\phi |11\rangle\right), \quad c_\phi \geq s_\phi > 0 \; . \tag{4.1.2}$$

Similarly for the other two possible combinations $|AB - C\rangle$ and $|AC - B\rangle$.

Tripartite Ent.

States which feature genuine tripartite entanglement between all the constituents. For these states, it holds that $r(\rho_A) = r(\rho_B) = r(\rho_C) = 2$. There are two inequivalent states for which this is true, the W class of states and the GHZ class of states. The details of these two states are discussed in what follows.

Recall that representatives of the two three-qubit states which can be considered maximally entangled are the so-called GHZ (after Greenberger, Horne and Zeilinger) or sometimes cat-state, given in equation (2.2.1), and the so-called W-state, given in equation (2.2.2). These two classes of states can be distinguished using the 3-tangle introduced in section 3.4.6. The 3-tangle vanishes for GHZ-type states, but does not vanish for W-type states.
These two types of states possess some very different features. Upon tracing out one of the constituent qubits of a GHZ-state, the remaining two qubits share a (mixed but) unentangled state. Tracing out one qubit from a W-state, however, the remaining two qubits will share an entangled state (Bell state). It is thus said that the GHZ-state features genuine three-way entanglement between all constituent qubits, while the W-state features two-way entanglement between all the 2-1 qubit pairs. What we mean by saying the entanglement between a 2-1 qubit pair is the entanglement between e.g. the pair of qubits BC and the individual qubit A, which can, despite the dimensional differences, be measured by the tangle $\tau_{A(BC)}$ as explained in [37].
Notice that instead of using the local rank as a measure to determine the entanglement classes of the different states, we could just as well have used the local entropies, as discussed earlier. A summary of three-qubit entanglement using the local entropies can be found in table 1.

Table 1: Summary of entanglement of three qubits

Class	Canonical Form	Measures			
		S_A	S_B	S_C	τ_3
A-B-C	$\lvert 000\rangle$	$=0$	$=0$	$=0$	$=0$
A-BC	$\lvert 0\rangle\,(c_\phi\,\lvert 00\rangle + s_\phi\,\lvert 11\rangle)$	$=0$	$\neq 0$	$\neq 0$	$=0$
AB-C	$(c_\phi\,\lvert 00\rangle + s_\phi\,\lvert 11\rangle)\,\lvert 0\rangle$	$\neq 0$	$\neq 0$	$=0$	$=0$
AC-B	$(c_\phi\,\lvert 00\rangle_{AC} + s_\phi\,\lvert 11\rangle_{AC})\,\lvert 0\rangle_B$	$\neq 0$	$=0$	$\neq 0$	$=0$
W	$\frac{1}{\sqrt{3}}\,(\lvert 001\rangle + \lvert 010\rangle + \lvert 100\rangle)$	$\neq 0$	$\neq 0$	$\neq 0$	$=0$
GHZ	$\frac{1}{\sqrt{3}}\,(\lvert 001\rangle + \lvert 010\rangle + \lvert 100\rangle)$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$

4.2 Four-Qubit Entanglement

Verstraete et al. analyse the different ways of entangling four-qubit states in [29], and evaluate some entanglement measures (including numerical generalisations of the 3-tangle to mixed states evaluated after one constituent qubit has been traced out) for representative states.
In this section we will summarise and comment on the findings by Verstraete et al. in [29].
What we consider here are *Families* of the orbits (3.3.2), where every state is transformed into a unique *normal form*, and if the normal form depends on *SLOCC invariants* it represents a family of orbits parametrised by the same. [38]
It should be noted that this classification in terms of families of orbits is distinct from the *entanglement classes* we used to distinguish states previously. In fact, the different families cannot be used to distinguish the states according to their entanglement, as a given family may contain both entangled and separable states. In addition, there is no mechanism to easily find the family a given state belongs to. A finer set of families that distinguish unambiguously between states based on the amount of entanglement they contain, as well as an easy way to ascribe a family to a given state, are desirable.
In detail, the 9 families obtained by Verstraete et al. for the four-qubit case are[1]

[1] In this formula, the last two signs in the last term of L_{ab_3} have been changed, to correct for a misprint in the set as given in [29], as pointed out by Chterental and Đoković in [39]. This change also unifies this set with the one given in [38].

$$
\begin{aligned}
G_{abcd} &= \frac{a+d}{2}\left(|0000\rangle + |1111\rangle\right) + \frac{a-d}{2}\left(|0011\rangle + |1100\rangle\right) \\
&+ \frac{b+c}{2}\left(|0101\rangle + |1010\rangle\right) + \frac{b-c}{2}\left(|0110\rangle + |1001\rangle\right) \\
L_{abc_2} &= \frac{a+b}{2}\left(|0000\rangle + |1111\rangle\right) + \frac{a-b}{2}\left(|0011\rangle + |1100\rangle\right) + c\left(|0101\rangle + |1010\rangle\right) + |0110\rangle \\
L_{a_2b_2} &= a\left(|0000\rangle + |1111\rangle\right) + b\left(|0101\rangle + |0101\rangle + |1010\rangle\right) + |0110\rangle + |0011\rangle \\
L_{ab_3} &= a\left(|0000\rangle + |1111\rangle\right) + \frac{a+b}{2}\left(|0101\rangle + |1010\rangle\right) + \frac{a-b}{2}\left(|0110\rangle + |1001\rangle\right) \\
&+ \frac{i}{\sqrt{2}}\left(|0001\rangle + |0010\rangle - |0111\rangle - |1011\rangle\right) \\
L_{a_4} &= a\left(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle\right) + i\,|0001\rangle + |0110\rangle - i\,|1011\rangle \\
L_{a_2 0_{3\oplus\bar{1}}} &= a\left(|0000\rangle + |1111\rangle\right) + |0011\rangle + |0101\rangle + |0110\rangle \\
L_{0_{5\oplus\bar{3}}} &= |0000\rangle + |0101\rangle + |1000\rangle + |0111\rangle \\
L_{0_{7\oplus\bar{1}}} &= |0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle \\
L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}} &= |0000\rangle + |0111\rangle \ ,
\end{aligned}
$$

where the parameters a, b, c, d are the four *SLOCC* invariants. The authors also note the families some commonly seen states belong to. For example, a state that is completely separable is part of the family L_{abc_2} if we set $a = b = c = d = 0$, and a state that is partially separable, consisting of an EPR pair combined with two unentangled qubits belongs to the family $L_{a_2b_2}$ with the invariants $a = b = 0$. If we combine two EPR pairs into a state, it is part of G_{abcd} with $a = 1; b = c = d = 0$ or $a = b = c = d$. Combining a 3-qubit GHZ-state with one separable qubit leads to a state in the family $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$, and similarly if one uses a 3-qubit W-state it belongs to $L_{a_2 0_{3\oplus\bar{1}}}$ with $a = 0$. The 4-qubit W-state given by

$$
|W\rangle_4 = \frac{1}{2}\left(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle\right) \ , \tag{4.2.1}
$$

belongs to L_{ab_3} where $a = b = 0$.
When looking at the 4-qubit GHZ-state given by

$$
|GHZ\rangle_4 = \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right) \ , \tag{4.2.2}
$$

we see that this state belongs to the family G_{abcd} if we let $a = d, b = c = 0$. We observe that the family L_{abc_2} contains both separable and maximally entangled states, as also noted in [38]. As mentioned previously, this is the reason that these families cannot be interpreted in the same way as the *entanglement classes* previously used to distinguish inequivalently entangled states.
We will only be concerned with *nilpotent* orbits, i.e. orbits under the *SLOCC* equivalence group, given by the quotient group as explained in section 3.3, for

which all invariants are equal to zero. For this special set, a hierarchy was proposed in [38], which can be seen from figure 2.
Looking at the values of the four-qubit generalisation of the 3-tangle (3.4.28) and the concurrence (3.4.27) for these states, it is noted that both these quantities are equal to zero for all states in the hierarchy in figure 2. In addition, this is also true for the four-qubit W-state (4.2.1).
Interestingly, for the four-qubit GHZ-state (4.2.2), we find for the four-qubit generalisation of the 3-tangle and the concurrence that $\tau_4 = C_4 = 1$. The tensor product of two EPR pairs has $\tau_4 = \frac{1}{\sqrt{2}}$ and $C_4 = 1$, despite not containing any genuine four-qubit entanglement, as was also observed in [31]. This suggests that these two measures are insufficient to make a meaningful statement about the entanglement of four-qubit states.
Additionally, we saw that the three-qubit classification necessitated the introduction of the 3-tangle, which, apart from just being an entanglement monotone, is also an *SLOCC* covariant. This is part of the motivation for the discussion that follows, in which we attempt a classification of entanglement using *SLOCC* invariants and covariants. For the three-qubit case this has previously been done successfully, while the same for the case of four-qubit systems is an open problem.

5 Covariant *SLOCC* Classification of Quantum Entanglement

In the previous section we have seen that, while for two and three qubit systems the *SLOCC* classification is quite manageable, for four or more qubits it becomes increasingly complicated. To go further, a more systematic, manifestly *SLOCC covariant* treatment is required.
One approach is to consider the minimal set of algebraically independent *SLOCC* invariant and covariant quantities, as mentioned in section 3.3.
The idea is then to analyse the conditions the vanishing of these quantities imposes on the states, and then to see which state in the hierarchy a general state that fulfils these conditions is *SLOCC* equivalent to. This approach will be used for a classification of three and four-qubit entanglement in this section. For three-qubit systems this has previously been done successfully and is summarised in section 5.1. In the four-qubit case this is an open problem, that has been partially solved in the literature for a specific subset of states living in the nilpotent orbits of the hierarchy shown in figure 2, with further constraints. Lifting these further constraints, and generalising the classification to all states in the nilpotent orbits is the subject of section 5.2.
To achieve this, in general we are interested in quantities invariant or covariant under the *SLOCC* group action. Invariants don't change under the application of an action of the group (or equivalently transform as the singlet representation

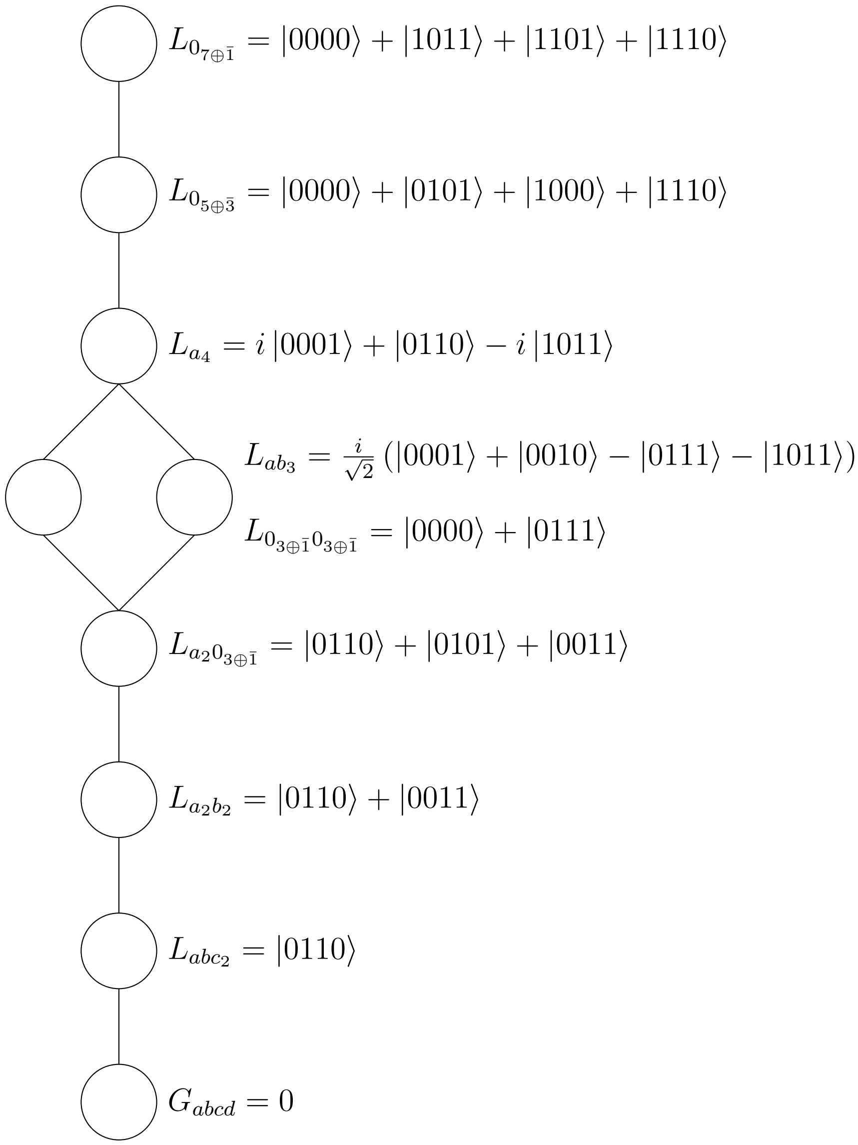

Figure 2: Hierarchy of 4-qubit families with all invariants set to zero as given in [38]. Note that normalisations have been omitted here.

of that group), whereas covariants transform appropriately, according to one of the representations of the group that is not the singlet.

5.1 Covariant Classification for Three Qubits

We have seen the conventional classification of three-qubit entanglement in section 4.1. The quantities used there to classify three-qubit entanglement are not manifestly $SLOCC$ invariant. While the three-tangle happens to, in fact, be $SLOCC$ invariant, the local entropies are only $LOCC$ invariants. [40]
The classification of three-qubit entangled states can also be constructed explicitly using invariant and covariant quantities. This was done by the authors in [40], and is summarised here as a prelude to the discussion on the four-qubit entanglement classification, which is the main focus of this section.
Note that not all invariants that can be constructed for the $SLOCC$ group action $SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ are necessary for the entanglement classification. First, it is useful to note that one of the $SL(2,\mathbb{C})$ invariant tensors is ε_{AB}, which follows from $\varepsilon_{AB} \mapsto {S_A}^M {S_B}^N \varepsilon_{MN} = \det(S)\varepsilon_{AB}$ (where the summation is implicit) by properties of the Levi-Civita Tensor, and $\det S = 1$ as we are considering $S \in SL(2,\mathbb{C})$.
In order to be consistent with the treatment of the four-qubit case in the following section, we adopt the notation given in [41].
Writing the Quantum State as a trilinear form

$$|\Psi\rangle = \sum_{ijk} a_{ijk} w_i x_j y_k \ , \tag{5.1.1}$$

the invariants and covariants can be written as polynomials in the coefficients a_{ijk} and variables w_i, x_j, y_k. The convention is to use a letter X^m_{pqr} to denote the covariants, where the X indicates the degree in the coefficients a_{ijk} (where A = 1, B = 2, ...) and the subscripts p,q,r indicate the degree in the variables w, x, y. The superscript serves to distinguish covariants that otherwise have the same properties and would be represented by the same symbol using the notation introduced above.
The authors in [40] construct three covariants quadratic in the coefficients. To this end, note that a covariant quantity quadratic in a_{ijk} transforms as a representation contained in

$$(\mathbf{2},\mathbf{2},\mathbf{2}) \times (\mathbf{2},\mathbf{2},\mathbf{2}) = (\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1}) \ , \tag{5.1.2}$$

such that we have dim = 64 on both sides.
The right hand side $(\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1})$ includes all combinations, e.g. $(\mathbf{3},\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{3},\mathbf{3}), (\mathbf{1},\mathbf{1},\mathbf{1})$ and so forth.
The covariants quadratic in the coefficients constructed by the authors (in their notation called $\gamma^{\square}$), expressed in the notation introduced above, are then

$$\begin{aligned}
B_{200} &= \varepsilon^{B_1B_2}\varepsilon^{C_1C_2}a_{A_1B_1C_1}a_{A_2B_2C_2}w^{A_1}w^{A_2} = \left(\gamma^A\right)_{A_1A_2}w^{A_1}w^{A_2}, \\
B_{020} &= \varepsilon^{C_1C_2}\varepsilon^{A_1A_2}a_{A_1B_1C_1}a_{A_2B_2C_2}x^{B_1}x^{B_2} = \left(\gamma^B\right)_{B_1B_2}x^{B_1}x^{B_2}, \\
B_{002} &= \varepsilon^{A_1A_2}\varepsilon^{B_1B_2}a_{A_1B_1C_1}a_{A_2B_2C_2}y^{C_1}y^{C_2} = \left(\gamma^C\right)_{C_1C_2}y^{C_1}y^{C_2} ,
\end{aligned} \tag{5.1.3}$$

where $\gamma^A, \gamma^B, \gamma^C$ is the original notation used by the authors which will be useful in constructing other covariants.
These transform as the $(\mathbf{3},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{3})$ respectively, and are directly related to the local entropies.
Also necessary for a classification is a covariant cubic in the coefficients. This covariant is the triple product, which maps a state into another, both transforming as a $(\mathbf{2},\mathbf{2},\mathbf{2})$, and is given by

$$T\left(\Psi,\Psi,\Psi\right) = T_{ABC}w^Ax^By^C \ . \tag{5.1.4}$$

It can be defined in three equivalent ways in terms of the $\gamma^{\square}$ given in equation (5.1.3) as

$$\begin{aligned}
C_{111} &= \varepsilon^{A_1A_2}a_{A_1B_1C_1}\left(\gamma^A\right)_{A_2A_3}w^{A_3}x^{B_1}y^{C_1} = T_{A_3B_1C_1}w^{A_3}x^{B_1}y^{C_1} \\
&= \varepsilon^{B_1B_2}a_{A_1B_1C_1}\left(\gamma^B\right)_{B_2B_3}w^{A_1}x^{B_3}y^{C_1} = T_{A_1B_3C_1}w^{A_1}x^{B_3}y^{C_1} \\
&= \varepsilon^{C_1C_2}a_{A_1B_1C_1}\left(\gamma^C\right)_{C_2C_3}w^{A_1}x^{B_1}y^{C_3} = T_{A_1B_1C_3}w^{A_1}x^{B_1}y^{C_3} ,
\end{aligned} \tag{5.1.5}$$

where we have again adopted a different notation and related it to the T_{ABC} used by the authors in [40].
Finally, the last covariant necessary to complete a classification of three-qubit entanglement is actually an invariant. This is the quartic norm $q(\Psi)$, which is equivalent (up to pre-factors) to Cayley's Hyperdeterminant (and the 3-tangle) we've met earlier, as

$$q(\Psi) = -2\,\mathrm{Det}\,a_{ABC} \ . \tag{5.1.6}$$

These are all the necessary ingredients for the covariant three-qubit classification, which can be seen from table 2.

Table 2: Covariant Classification of Three-Qubit Entanglement

Class	Vanishing	Non-Vanishing
A-B-C	γ	Ψ
A-BC	$T(\Psi,\Psi,\Psi)$	γ^A
B-AC	$T(\Psi,\Psi,\Psi)$	γ^B
C-AB	$T(\Psi,\Psi,\Psi)$	γ^C
W	$q(\Psi)$	$T(\Psi,\Psi,\Psi)$
GHZ	–	$q(\Psi)$

An important feature of this classification is the hierarchical relation between the individual covariants. The vanishing of covariants as one moves up in the hierarchy (down in the table) implies the vanishing of all previous ones. For example, the vanishing of $T(\Psi,\Psi,\Psi)$ implies the vanishing of $q(\Psi)$, and the vanishing of all γ implies the vanishing of $T(\Psi,\Psi,\Psi)$. Note that we have to consider the vanishing of all γ simultaneously for this hierarchy to work, as the classes A-BC, B-AC and C-AB can be considered of the same rank in the hierarchy, as they contain the same entanglement up to permutations of A,B,C.
We have seen that the covariant classification is indeed feasible for the case of three qubits, and the generalisation to larger systems is the next step. In the following section, the same classification for a four-qubit system is discussed.

5.2 Covariant Classification for Four Qubits

We briefly discussed the *families* of entangled states as introduced in [29] and the difficulties that arise as each family could contain states with obviously different amounts of entanglement. This makes a finer classification given by conditions invariant under *SLOCC* / ILO desirable. Investigating which covariants of the *SLOCC equivalence* group vanish and which do not is one way of classifying different kinds of states. There are a minimum of 170 such covariants for the four-qubit case, which have been found in [41], and 4 invariant quantities [29, 38] as was pointed out previously.
As mentioned in section 4.1, in the three-qubit case this classification was equivalent to the classification using the local entropies and the 3-tangle. A logical next step is to attempt to obtain a similar classification for the four-qubit case. As per the arguments laid out previously about the *SLOCC* orbits in this case being parametrised continuously, this is more involved if one wishes to consider the parameters, given by the invariants, being free. As a starting point we will focus on nilpotent orbits, which are such that all invariants vanish. These states, and the proposed hierarchy taken from the Black Hole / Qubit correspondence, were given in figure 2. For this nilpotent subset, above mentioned difficulties, such as finding families containing states with manifestly different amounts of entanglement, don't arise at first sight. It should be noted that the states in the

hierarchy as given in the figure are *representatives* of these nilpotent orbits only. A classification will necessarily have to include all states that lie on nilpotent orbits, meaning that arbitrary coefficients in front of each term will be introduced. This will be further elaborated on in comparison to previous work in section 5.2.3.

5.2.1 General SLOCC Transformation

In order to analyse the different classes of entangled states it is helpful introduce a notation for the state coefficients and generating *SLOCC* transformations which is unaffected by permutations.
To this end, we adopt the notation introduced in [14] where

$$|\psi\rangle = \Phi\,|\phi\rangle \leftrightarrow \Phi = \{A_0, A_i, A_{ij}, A_{ijk}, A_{ijkl}\} \ , \tag{5.2.1}$$

where $i, j, k, l = 1, \ldots, 4$, such that

$$\left\{ A_0 = a_{0000}, A_i = \begin{pmatrix} a_{1000} \\ a_{0100} \\ a_{0010} \\ a_{0001} \end{pmatrix}, A_{ij} = \begin{pmatrix} 0 & a_{0011} & a_{0101} & a_{0110} \\ a_{0011} & 0 & a_{1001} & a_{1010} \\ a_{0101} & a_{1001} & 0 & a_{1100} \\ a_{0110} & a_{1010} & a_{1100} & 0 \end{pmatrix},\ A^i = \begin{pmatrix} a_{0111} \\ a_{1011} \\ a_{1101} \\ a_{1110} \end{pmatrix}, A^0 = a_{1111} \right\} \ , \tag{5.2.2}$$

and the raising / lowering of indices indicates a dualisation, i.e. $0 \leftrightarrow 1$.
Using these definitions, the SLOCC generating transformations are [14, 42]

$$\phi(C) : \begin{pmatrix} A_0 \\ A_i \\ A^{ij} \\ A^i \\ A^0 \end{pmatrix} \mapsto \begin{pmatrix} A_0 \\ C_i A_0 + A_i \\ d^{ijkl} C_k C_l A_0 + d^{ijkl} C_k A_l + A^{ij} \\ d^{ijkl} C_j C_k C_l A_0 + d^{ijkl} C_j C_k A_l + d^{ijkl} C_j A_{kl} + A^i \\ d^{ijkl} C_i C_j C_k C_l A_0 + d^{ijkl} C_i C_j C_k A_l + d^{ijkl} C_i C_j A_{kl} + d^{ijkl} C_i A_{jkl} + A^0 \end{pmatrix} \tag{5.2.3a}$$

$$\psi(D) : \begin{pmatrix} A_0 \\ A_i \\ A_{ij} \\ A^i \\ A^0 \end{pmatrix} \mapsto \begin{pmatrix} d_{ijkl} D^i D^j D^k D^l A^0 + d_{ijkl} D^i D^j D^k A^l + d_{ijkl} D^i D^j A^{kl} + d_{ijkl} D^i A^{jkl} + A_0 \\ d_{ijkl} D^j D^k D^l A^0 + d_{ijkl} D^j D^k A^l + d_{ijkl} D^j A^{kl} + A_i \\ d_{ijkl} D^k D^l A^0 + d_{ijkl} D^k A^l + A_{ij} \\ D^i A^0 + A^i \\ A^0 \end{pmatrix} , \tag{5.2.3b}$$

where $d_{ijkl} = |\varepsilon_{ijkl}|$ and C_i, D_i are the elements of the specific *SLOCC* transformation.
In the following section some examples of transformations using this notation are given, which will make their usage more apparent.

5.2.2 Covariant Classification of Four-Qubits

The covariants in the four-qubit case were found in [41]. We introduced the naming conventions used by the authors previously for the three-qubit case for sake of consistency, and recap it briefly here. Given that a four-qubit Quantum State $|\Psi\rangle$ can be written as a quadrilinear form

$$|\Psi\rangle = \sum_{i,j,k,l=0}^{1} a_{ijkl} w_i x_j y_k z_l \ , \tag{5.2.4}$$

the invariants and covariants can be written as polynomials in the coefficients a_{ijkl} and variables w_i, x_j, y_k, z_l.
Having introduced a general representation of $SLOCC$ transformations for four-qubits along with helpful notation in section 5.2.1, we can once more give some examples, where we use the same notation as previously, the covariants being given by letters indicating their degree in the coefficients together with subscripts indicating the degree in the variables w, x, y, z.
We note that the group of orbits given by

$$\frac{\mathbb{C}^{2^{\otimes 4}}}{[SL(2,\mathbb{C})]^{\times 4}} \tag{5.2.5}$$

is parametrised by 8 free parameters ($\mathbb{C}^{2^{\otimes 4}}$ having $2^4 = 16$ complex, 32 real degrees of freedom and $[SL(2,\mathbb{C})]^{\times 4}$ having $(2^2-1)^4 = 12$ complex, 24 real degrees of freedom, leaving 8 free real parameters). Thus these 4 complex parameters are the total of four algebraically independent invariants we can construct for this case. Those invariants are [41]

$$\left\{ B_{0000}, D^1_{0000}, D^2_{0000}, F_{0000} \right\} \ . \tag{5.2.6}$$

There is, in fact, a third quartic invariant (call it D^3) which is not linearly independent of the other two. [43] This invariant will become important later-on, when constructing manifestly permutation symmetric invariant quantities.
In order to construct an example of a covariant quantity, note that the product of the coefficients $a_{ABCD}a_{MNKL}$ transforms as a $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) \times (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$ under G, for which

$$(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) \times (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) = (\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1},\mathbf{3}+\mathbf{1}) \ , \tag{5.2.7}$$

where we have dim = 256 on both sides. The right hand side in equation (5.2.7) again includes all combinations, e.g. $(\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1}), (\mathbf{1},\mathbf{3},\mathbf{3},\mathbf{3}), (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$ and so forth. Contracting all indices of a quantity of second degree in the coefficients a_{ijkl} leads to the construction of the invariant B_{0000}

$$B_{0000} = a_{ABCD}a_{MNKL}\ \varepsilon^{AM}\varepsilon^{BN}\varepsilon^{CK}\varepsilon^{DL} \ , \tag{5.2.8}$$

which, up to pre-factors, is equivalent to the Hyperdeterminant discussed earlier, and as an invariant transforms as the $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$.
An example of a covariant quantity of second degree can be built similarly,

$$B_{0022} = a_{AB(CD)}a_{MN(KL)}\ \varepsilon^{AM}\varepsilon^{BN}y^C y^K z^D z^L\ , \tag{5.2.9}$$

which corresponds to the representation $(\mathbf{1},\mathbf{1},\mathbf{3},\mathbf{3})$, and the symmetrisation in the indices is necessary as the $\mathbf{3}$ is a symmetric tensor. As the *SLOCC* group action also contains a re-shuffling of the individual qubits (e.g. $A \leftrightarrow B$), we are not normally interested to differentiate permutation symmetric states and treat a collection of permutation symmetric covariants of the same type as a single covariant.

5.2.3 Covariant Classification of The Nilpotent Orbits of Four-Qubits

Consulting the full list of covariants in [41], we note that we can use them to impose conditions on our hierarchy of nilpotent states in figure 2.
A similar classification has already been performed for a subset of the states of interest in the present work in [1]. There, the authors consider a total of 11662 states which are states in the nilpotent orbits for which the coefficients a_{ijkl} as in equation (5.2.4) belong to the set $\{0,1\}$. This set essentially encompasses the representative state for each orbit up to the actions of local unitaries. Overall multiplication by complex factors does not change the states as they are considering the projective subset of all states in the Hilbert space. Knowing that all of these 11662 states will belong to one of the 8 orbits considered (as they are related by local unitaries acting on the representatives), the authors in [1] evaluate all covariants for each state and find patterns that uniquely identify which orbit a state belongs to. The results of their search are presented in table 3.
We aim to prove that this classification holds *in general* for all states in the nilpotent hierarchy, not just the subset of 11662 states considered by the authors, and the procedure to achieve this is outlined in this section.
As we are talking about nilpotent orbits, we impose that all invariants given in (5.2.6) vanish for the states considered.
We have then for the covariants

$$
\begin{aligned}
\mathbf{B} &= \{B_{2200}, B_{2020}, B_{2002}, B_{0220}, B_{0202}, B_{0022}\} \\
\mathbf{C} &= \{C_{3111}, C_{1311}, C_{1131}, C_{1113}\} \\
\mathbf{C}' &= \{C_{3111} \times C_{1311} \times C_{1131} \times C_{1113}\} \\
\mathbf{C}'' &= \left\{C^1_{1111}, C^2_{1111}\right\} \\
\mathbf{D} &= \{D_{4000}, D_{0400}, D_{0040}, D_{0004}\} \\
\mathbf{D}' &= \{D_{2200}, D_{2020}, D_{2002}, D_{0220}, D_{0220}, D_{0202}, D_{0022}\} \\
\mathbf{E} &= \left\{E^i_{3111}, E^i_{1311}, E^i_{1131}, E^i_{1113}\right\}, \quad i = 1, 2, 3 \\
\mathbf{F} &= \left\{F^1_{2220}, \text{and permutations}\right\} \\
\mathbf{F}' &= \{F_{4200}, \text{and permutations}\} \\
\mathbf{F}'' &= \{F_{2200}, \text{and permutations}\} \\
\mathbf{L} &= \{L_{6000}, L_{0600}, L_{0060}, L_{0006}\}\ ,
\end{aligned}
\tag{5.2.10}
$$

and the conditions the hierarchy imposes on these are given in table 3. It should be noted that the authors in [1] do not utilise the covariants $\mathbf{C}'', \mathbf{E}, \mathbf{F}', \mathbf{F}''$ in their classification, and we have thus far not found it necessary to consider them in this discussion either.

Table 3: Conditions on Covariants in Nilpotent State Hierarchy

Family	Canonical Form	Covariants						
		$\mathbf{B}$	$\mathbf{C}$	$\mathbf{C}'$	$\mathbf{D}$	$\mathbf{D}'$	$\mathbf{F}$	$\mathbf{L}$
L_{abc_2}	A-B-C-D	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$
$L_{a_2b_2}$	A-B-EPR	$\neq 0$	$=0$	$=0$	$=0$	$=0$	$=0$	$=0$
$L_{a_20_{3\oplus\bar{1}}}$	A-W	$\neq 0$	$\neq 0$	$=0$	$=0$	$=0$	$=0$	$=0$
$L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$	A-GHZ	$\neq 0$	$\neq 0$	$=0$	$\neq 0$	$=0$	$=0$	$=0$
L_{ab_3}	L_{ab_3}	$\neq 0$	$\neq 0$	$\neq 0$	$=0$	$=0$	$=0$	$=0$
L_{a_4}	L_{a_4}	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$=0$	$=0$
$L_{0_{5\oplus\bar{3}}}$	$L_{0_{5\oplus\bar{3}}}$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$=0$
$L_{0_{7\oplus\bar{1}}}$	$L_{0_{7\oplus\bar{1}}}$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$

Note further that the vanishing of lower order covariants implies the vanishing of higher order covariants. For example, the vanishing of $\mathbf{B}$ also implies the vanishing of $\mathbf{C}, \mathbf{D}, \ldots$. Conversely, when going downwards in table 3, which corresponds to going upwards in the hierarchy, if a covariant was non-zero in the previous case, it remains non-zero as one proceeds downwards in the table. For example, for $L_{a_20_{3\oplus\bar{1}}}$ we still have that $\mathbf{B} \neq 0$. Note that the exception to these rules is $\mathbf{C}'$.

In order to verify that this classification also holds for a general state that is part of the nilpotent hierarchy, evaluating the conditions the vanishing of a covariant imposes on a general four-qubit state, and then checking that these conditions imposed on a general expression for a higher order covariant make it vanish is the first step. The second step is to find the conditions the vanishing of the covariants imposes on a general state, and then showing that a general state that satisfies these conditions is indeed *SLOCC* equivalent to the corresponding canonical representative state in the hierarchy.
The introduction of the product covariant $\mathbf{C}'$, which is manifestly permutation symmetric, suggests that one might also look at other manifestly permutation symmetric quantities. In order to make the permutation symmetry of the invariants manifest, one proceeds as described in [43], such that

$$\begin{aligned} \mathbf{I^2} &= \mathbf{B_{0000}} \\ \mathbf{I^6} &= \mathbf{F_{0000}} \\ \mathbf{I^8} &= M^2 + N^2 + P^2 \\ \mathbf{I^{12}} &= (M-N)(N-P)(P-M) \\ \text{where} \quad M &= \mathbf{D^1} - \mathbf{D^2}, \quad N = \mathbf{D^2} - \mathbf{D^3}, \quad P = \mathbf{D^3} - \mathbf{D^1} \ , \end{aligned} \tag{5.2.11}$$

and we have omitted the subscript 0000 of $\mathbf{D^1}, \mathbf{D^2}, \mathbf{D^3}$. $\mathbf{I^2}, \mathbf{I^6}, \mathbf{I^8}, \mathbf{I^{12}}$ are the manifestly permutation symmetric invariants.
To proceed with the proof of the covariant classification for the general case, yet another notation needs to be introduced. Denoting the coefficients to the state by a_{ijkl}, such that

$$|\Psi\rangle = a_{ijkl} \, |ijkl\rangle \ , \tag{5.2.12}$$

where the sum is implicit and $i, j, k, l = 0, 1$, we can also denote the coefficients by a decimal number, rather than binary, such that

$$a_{ijkl} = b_m, \quad m = 8i + 4j + 2k + l \ . \tag{5.2.13}$$

Using this notation, it is first noted that a general four-qubit state can, without loss of generality, be *SLOCC* transformed into a *Reduced State*, for which

$$|\Psi_{red}\rangle \coloneqq b_7 \mapsto 0, b_{11} \mapsto 0, b_{13} \mapsto 0, b_{14} \mapsto 0, b_{15} \mapsto 1 \ , \tag{5.2.14}$$

and all other b remain arbitrary.
The computations in what follows are most conveniently done using a computer algebra system.

A-B-C-D (L_{abc_2}) $\quad \mathbf{A} \neq 0$, $\mathbf{B} = 0$
In the case of the separable state, we find immediately that the vanishing of $\mathbf{B}$ imposes that the *Reduced State* (5.2.14) be written as $b_{13}\,|1101\rangle + |1111\rangle$, which is a separable state. More formally, it can be transformed into $|1111\rangle$ using the transformation $\psi(D)$, given in equation (5.2.3b), with $D = (0, 0, -b_{13}, 0)$. The vanishing of $\mathbf{B}$ also implies the vanishing of $\mathbf{C}$ as required.

A-B-EPR $(L_{a_2b_2})$ $\quad \mathbf{B} \neq 0$,$\mathbf{C} = 0$
Imposing the vanishing of $\mathbf{C}$ leads immediately to the condition on the reduced state that $b_1, b_2, b_4, b_8 = 0$. Reapplying these conditions to the expression for $\mathbf{C}$, and ensuring the vanishing of $\mathbf{I^2}$ and the non-vanishing of $\mathbf{B}$, one then finds that at least one of $\{b_3, b_5, b_6, b_9, b_{10}, b_{12}\}$ is not equal to zero. As the permutation group acts transitively on this set, we can choose, say, $b_3 \neq 0$ without loss of generality. Making this choice then implies that the rest of the set vanish, leaving the reduced state as $b_3\,|0011\rangle + |1111\rangle$, which is indeed tri-separable as required.

A-W $(L_{a_2 0_{3\oplus\bar{1}}})$ $\quad C_{1113} \neq 0$, $C_{3111} = 0$, $C_{1311} = 0$, $C_{1131} = 0$, $\mathbf{D} = 0$
For this class we know that both $\mathbf{B}$ and $\mathbf{C}$ are non-vanishing, while $\mathbf{C}'$ vanishes. Notice, however, that both $\mathbf{C}$ and $\mathbf{C}'$ are made up of the same components, added together or multiplied together respectively. Thus, to satisfy these constraints, it is enough to assume that one component of $\mathbf{C}$ is non-vanishing. Say we choose C_{1113} to be non-vanishing. It is then helpful to notice that certain permutations of the i, j, k, l in the coefficients a_{ijkl} leave C_{1113} invariant. It can be verified that these permutations rotate the remaining components of $\mathbf{C}$ into each other, but never into C_{1113}, such that we can apply these permutations without loss of generality. It should also be noted that D_{0004} is left invariant under these permutations, while the other components of $\mathbf{D}$ are rotated into each other. From the non-vanishing of C_{1113} and $\mathbf{I^2}$ it follows that $b_0, b_2, b_4, b_8 = 0$. We then find that, $b_{10}b_{12} = 0$, where we can choose, say $b_{10} = 0$ since as part of our allowed permutations we can rotate b_{10} and b_{12} into each other. Similarly, for $b_6 b_{12} = 0$, which we can also rotate into each other using allowed permutations that leave b_{10} invariant, and choose, say $b_6 = 0$. Then from $b_3 b_{12} = 0$ both these possibilities have to be considered. In the case of $b_{12} = 0$, the reduced state becomes

$$\begin{aligned} |\psi_{\text{red}}\rangle &= b_1\,|0001\rangle + b_3\,|0010\rangle + b_5\,|0101\rangle + b_9\,|1001\rangle + |1111\rangle \\ &= \Big(\underbrace{b_1\,|000\rangle + b_3\,|001\rangle + b_5\,|010\rangle + b_9\,|100\rangle + |111\rangle}_{(\diamond)}\Big) \otimes |1\rangle \;, \end{aligned} \tag{5.2.15}$$

which is bi-separable. Looking only at the A,B and C qubits, given in $(\diamond)$, (regarding it as a three-qubit state), it is apparent that, up to normalisation, this is a three-qubit GHZ-state added to a three-qubit W-state. To find out its entanglement properties, we evaluate the Hyperdeterminant or 3-tangle of ($\diamond$), which is given by

$$\mathrm{Det}\, a = a_0^2 a_7^2 + 4a_1 a_2 a_4 a_7 = b_1^2 + 4b_3 b_5 b_9 \ . \tag{5.2.16}$$

But we know the result of this, as this is exactly the expression we are left with for D_{0004}, as all other components of $\mathbf{D}$ have vanished at this point. Hence we know it must be zero, making the Hyperdeterminant equal to zero, making ($\diamond$) indeed a W-type state, such that the total reduced state is of the form W-D.
Considering the other case, choosing $b_3 = 0$ rather than b_{12}, implies from $\mathbf{D}$ vanishing but C_{1113} non-vanishing that $b_{12} = 0$ and $b_5, b_9 \neq 0$, and the reduced state then reads

$$\begin{aligned} |\psi_{\text{red}}\rangle &= b_5 |0101\rangle + b_9 |1001\rangle + |1111\rangle \\ &= \Big(\underbrace{b_5 |010\rangle + b_9 |100\rangle + |111\rangle}_{(\diamond\diamond)} \Big) \otimes |1\rangle \ , \end{aligned} \tag{5.2.17}$$

which is again clearly bi-separable, and ($\diamond\diamond$) is manifestly of the three-qubit W-type, which can be seen more easily after bit-flipping the third qubit.
If one imposes that another one of the components of the $\mathbf{C}$ is non-vanishing from the beginning, say C_{3111}, one is indeed left with another state of the same type, in this case A-W.

A-GHZ ($L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$) $D_{0004} \neq 0$, $D_{4000} = 0$, $D_{0400} = 0$, $D_{0040} = 0$
This case proceeds nearly exactly as the A-W case discussed previously. At the point where we arrive at the branching $b_3 = 0$ or $b_{12} = 0$, considering $b_{12} = 0$ we arrive at the same reduced state

$$\begin{aligned} |\psi_{\text{red}}\rangle &= b_1 |0001\rangle + b_3 |0010\rangle + b_5 |0101\rangle + b_9 |1001\rangle + |1111\rangle \\ &= \Big(b_1 |000\rangle + b_3 |001\rangle + b_5 |010\rangle + b_9 |100\rangle + |111\rangle \Big) \otimes |1\rangle \ . \end{aligned} \tag{5.2.18}$$

The difference is that in this case $\mathbf{D}$ is non-vanishing rather than vanishing, from which it follows that the Hyperdeterminant, given by

$$\mathrm{Det}\, a = a_0^2 a_7^2 + 4a_1 a_2 a_4 a_7 = b_1^2 + 4b_3 b_5 b_9, \tag{5.2.19}$$

is non-vanishing rather than vanishing. The state is thus indeed of the GHZ-D type.
Considering the other branch, where $b_3 = 0$, we find from the non-vanishing of $\mathbf{D}$ that $b_1 \neq 0$. It follows that $b_{12} = 0$ and $b_5, b_9 \neq 0$, and the reduced state becomes

$$\begin{aligned}|\psi_{\text{red}}\rangle &= b_1\,|0001\rangle\, b_5\,|0101\rangle + b_9\,|1001\rangle + |1111\rangle \\ &= \Big(\underbrace{b_1\,|000\rangle + b_5\,|010\rangle + b_9\,|100\rangle + |111\rangle}_{(\diamond\diamond\diamond)}\Big) \otimes |1\rangle \quad . \end{aligned} \tag{5.2.20}$$

The Hyperdeterminant evaluated for $(\diamond \diamond \diamond)$ reduces further than in the previous cases, to

$$\operatorname{Det} a = a_0^2 a_7^2 = b_1^2, \tag{5.2.21}$$

which is non-zero, such that again the state is of the GHZ-W type.
Just as in the W-A case it also holds here that if one imposes that another one of the components of the $\mathbf{C}$ is non-vanishing from the beginning, say C_{3111}, one is again left with another state of the same type, in this case A-GHZ.
Note that the choice of which component of the $\mathbf{C}$ is non-vanishing is also the deciding factor in the end which single component of the $\mathbf{D}$ remains as non-vanishing.

Continuing in The Hierarchy
The work on the rest of the classification is ongoing.

5.3 Experimental Verification of The Classification

In addition to the theoretical verification of the entanglement hierarchy of the nilpotent four-qubit states as pursued in the previous section, it is desirable to also verify this hierarchy of entangled states in an experiment. The idea of a Quantum Games provides a possible approach to such an experiment that would allow us to distinguish between them, as was done for the two and three-qubit case in sections 2.1.3 and 2.2. A game that reflects the hierarchy in terms of improving the players' chance of winning (collectively or individually) more the higher up in the hierarchy their shared state is would be suited to that purpose. Four-Player Quantum Games can be constructed in the same way we constructed the Three-Player Games in section 2.2, if we consider the four-qubit equivalents of the maximally entangled states, i.e. the W-type and the GHZ-type. However, what we are interested in is constructing a game that reflects the hierarchy introduced in figure 2, which proves more difficult. The standard strategies employed to construct the three-player games do not lead to similar advantages using these states, often even performing worse than a classical strategy. Changing the question set, or the requirements on the answers alone does not solve this problem. In fact, introducing larger question sets with more complicated requirements on the answers given by the players leads to other complications, as it increases the difficulty of obtaining and proving the optimal classical strategy and its performance. There are other types of games to consider which serve well to establish the performance of the GHZ- and W-states in smaller systems, such as the Magic Square Game or Kochen-Specker Game, see for example [44]. However, while these games can be generalised to larger systems, they still remain geared towards performing well when the parties share a GHZ-type state, and therefore fail to distinguish the other forms of entanglement encountered in the *SLOCC* classification. Another potential route to finding the desired games is to consider not just the case of cooperative games as the examples above, but also competitive games, opening up the area of Quantum Game Theory. Just as in the games introduced above it can be advantageous in Quantum Game Theory for the parties to share an appropriate entangled state, see [45] for a simple example and [46] for a general explanation of the analogy between aspects of Game Theory and Quantum Games. However, we face the same problem that these games, for which a Quantum Advantage becomes apparent when a GHZ-state or combinations of EPR pairs are utilised, are not straight-forwardly generalisable to show similar behaviour when the parties share states from the nilpotent hierarchy.

6 Conclusion

We have reported on a first attempt at generalising the existing classification of four-qubit states living on nilpotent orbits under the paradigm of *SLOCC Equivalence*. This attempt was successful for the separable, bi-separable and tri-separable cases, while the work on non-separable states is ongoing. The success in those cases is very promising however, so that the approach of the *SLOCC* covariant classification of entanglement, which has been proven to work well in the three-qubit case, seems to indeed be generalisable to the four-qubit case, at least for nilpotent orbits.

Of course, much more work on the topic is needed. The classification for the nilpotent four-qubit states needs to be finalised. In addition, a scheme that can be implemented experimentally to verify the hierarchy is highly desirable. It is then still a long way from the classification of nilpotent orbits to a general classification, valid for all possible four-qubit states. In this case, the structure of entanglement classes is far more intricate. As an example, it has been shown that there exists an ADE-type correspondence between the *SLOCC* orbits and simple singularities of type D_4. [47] Looking further into the future, obtaining a similar classification for even larger systems with more than four qubits is the next step. This is likely to be a complicated undertaking, as the landscape of *SLOCC equivalence* orbits and entangled states even with just five qubits will be far richer than that of four qubits. Clearly, a more computationally efficient or qualitative approach is needed for larger systems. However, as it stands we have only one non-trivial data point, the three-qubit classification. In this regard a complete understanding of the four-qubit case will provide much needed guidance in any attempts to generalise the classification.

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